**Differential Geometry of Framed Curves**
PROF. JOHN MADDOCKS
SESSION 4: SOLUTIONS T. LESSINNES

Before we start, we do the following computation.

Let $\boldsymbol{\alpha}(t)$ be a $\mathcal{C}^1(I, \mathbb{R}^3 \setminus \{0\})$ function and $\boldsymbol{e} = \boldsymbol{\alpha}/||\boldsymbol{\alpha}||$. We then have

$$\frac{d\boldsymbol{e}}{dt} = \frac{d}{dt}\frac{\boldsymbol{\alpha}}{||\boldsymbol{\alpha}||} = \frac{||\boldsymbol{\alpha}||\,\boldsymbol{\alpha}' - \frac{\boldsymbol{\alpha}\cdot\boldsymbol{\alpha}'}{||\boldsymbol{\alpha}||}\,\boldsymbol{\alpha}}{||\boldsymbol{\alpha}||^2} = \frac{1}{||\boldsymbol{\alpha}||}\left(\boldsymbol{\alpha}' - (\boldsymbol{e}\cdot\boldsymbol{\alpha}')\,\boldsymbol{e}\right) = \boldsymbol{e}\times\left(\frac{\boldsymbol{\alpha}'}{||\boldsymbol{\alpha}||}\times\boldsymbol{e}\right). \tag{1}$$

# 1 Link as a signed area

Since $\mathbf{e}(\sigma, s) = \frac{\mathbf{y}(\sigma)-\mathbf{x}(s)}{||\mathbf{y}(\sigma)-\mathbf{x}(s)||}$, by the computation at the beginning of this solutions sheet we have

$$\mathbf{e}_\sigma = \frac{1}{||\mathbf{y}-\mathbf{x}||}(\mathbf{y}_\sigma - (\mathbf{y}_\sigma\cdot\mathbf{e})\mathbf{e})$$

and

$$\mathbf{e}_s = \frac{-1}{||\mathbf{y}-\mathbf{x}||}(\mathbf{x}_s - (\mathbf{x}_s\cdot\mathbf{e})\mathbf{e})$$

this means that

$$\begin{aligned}
\mathbf{e}\cdot(\mathbf{e}_s\times\mathbf{e}_\sigma) &= \mathbf{e}\cdot\left(\frac{-1}{||\mathbf{y}-\mathbf{x}||}(\mathbf{x}_s - (\mathbf{x}_s\cdot\mathbf{e})\mathbf{e}) \quad\times\quad \frac{1}{||\mathbf{y}-\mathbf{x}||}(\mathbf{y}_\sigma - (\mathbf{y}_\sigma\cdot\mathbf{e})\mathbf{e})\right)\\
&= \mathbf{e}\cdot\frac{-1}{||\mathbf{y}-\mathbf{x}||^2}\left(\mathbf{x}_s\times\mathbf{y}_\sigma - (\mathbf{y}_\sigma\cdot\mathbf{e})\mathbf{x}_s\times\mathbf{e} - (\mathbf{x}_s\cdot\mathbf{e})\mathbf{e}\times\mathbf{y}_\sigma\right)\\
&= \frac{1}{||\mathbf{y}-\mathbf{x}||^2}\,\mathbf{e}\cdot(\mathbf{y}_\sigma\times\mathbf{x}_s) = \frac{(\mathbf{y}(\sigma)-\mathbf{x}(s))\cdot(\mathbf{y}'(\sigma)\times\mathbf{x}'(s))}{||\mathbf{y}(\sigma)-\mathbf{x}(s)||^3}.
\end{aligned}$$

Now, given a survace $\mathbf{r}(x, y)$ with normal vector $\mathbf{n_r}(x, y)$, the signed survace area of a patch $S$ of $\mathbf{r}$ is given by

$$\Sigma = \iint_S \mathbf{n_r}\cdot(\mathbf{r}_x\times\mathbf{y}_y)\;dx\,dy.$$

In this case, since $\mathbf{e}$ spans (a portion of) the unit sphere, we have that $\mathbf{n_e} = \mathbf{e}$, and the statement follows.

# 2 Homotopy invariance

Under the variations

$$\mathbf{x}(s;\varepsilon) = \mathbf{x}(s) + \varepsilon\,\delta\mathbf{x}(s) \qquad \text{and} \qquad \mathbf{y}(\sigma;\varepsilon) = \mathbf{y}(\sigma) + \varepsilon\,\delta\mathbf{y}(\sigma),$$

we have

$$\mathbf{e}(s,\sigma;\varepsilon) = \frac{\mathbf{y}(\sigma) - \mathbf{x}(s) + \varepsilon\,(\delta\mathbf{y}(\sigma) - \delta\mathbf{x}(s))}{||\mathbf{y}(\sigma) - \mathbf{x}(s) + \varepsilon\,(\delta\mathbf{y}(\sigma) - \delta\mathbf{x}(s))||},$$

and therefore by (1), we have

$$\frac{d\mathbf{e}}{d\varepsilon} = \frac{\mathbf{e}(s,\sigma;\varepsilon)\times\big((\delta\mathbf{y}(\sigma) - \delta\mathbf{x}(s))\times\mathbf{e}(s,\sigma;\varepsilon)\big)}{||\mathbf{y}(\sigma) - \mathbf{x}(s) + \varepsilon\,(\delta\mathbf{y}(\sigma) - \delta\mathbf{x}(s))||}.$$

Then for $\varepsilon = 0$,

$$\delta \mathbf{e} = \frac{1}{||\mathbf{y}-\mathbf{x}||}\Big(\mathbf{e} \times \big((\delta\mathbf{y} - \delta\mathbf{x}) \times \mathbf{e}\big)\Big).$$

The variation $\delta$Lk is then

$$\begin{aligned} 4\pi\,\delta\mathrm{Lk} &= \delta \int_0^{l_1}\int_0^{l_2} \mathbf{e}\cdot(\mathbf{e}_\sigma \times \mathbf{e}_s)\; d\sigma\, ds = \int_0^{l_1}\int_0^{l_2} \delta\big(\mathbf{e}\cdot(\mathbf{e}_\sigma \times \mathbf{e}_s)\big)\; d\sigma\, ds \\ &= \int_0^{l_1}\int_0^{l_2} \delta\mathbf{e}\cdot(\mathbf{e}_\sigma \times \mathbf{e}_s) + \mathbf{e}\cdot(\delta\mathbf{e}_\sigma \times \mathbf{e}_s) + \mathbf{e}\cdot(\mathbf{e}_\sigma \times \delta\mathbf{e}_s)\; d\sigma\, ds. \end{aligned}$$

Now, we have that

$$\begin{aligned} \int_0^{l_2} \mathbf{e}\cdot(\delta\mathbf{e}_\sigma \times \mathbf{e}_s)\; d\sigma &= \int_0^{l_2} \delta\mathbf{e}_\sigma\cdot(\mathbf{e}_s \times \mathbf{e})\; d\sigma \\ &= \underbrace{\Big[\delta\mathbf{e}\cdot(\mathbf{e}_s \times \mathbf{e})\Big]_0^{l_2}}_{=0 \text{ since all these are periodic}} - \int_0^{l_2} \delta\mathbf{e}\cdot\frac{d}{d\sigma}(\mathbf{e}_s \times \mathbf{e})\; d\sigma \\ &= -\int_0^{l_2} \delta\mathbf{e}\cdot(\mathbf{e}_{s\sigma} \times \mathbf{e}) + \delta\mathbf{e}\cdot(\mathbf{e}_s \times \mathbf{e}_\sigma)\; d\sigma. \end{aligned}$$

Similarly,

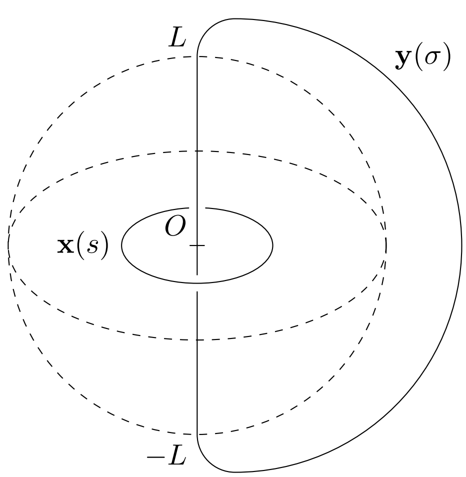
$$\int_0^{l_1} \mathbf{e}\cdot(\mathbf{e}_\sigma \times \delta\mathbf{e}_s)\; ds = -\int_0^{l_1} \delta\mathbf{e}\cdot(\mathbf{e}_s \times \mathbf{e}_\sigma) + \delta\mathbf{e}\cdot(\mathbf{e} \times \mathbf{e}_{\sigma s})\; d\sigma \qquad (2)$$

Hence,

$$\delta\mathrm{Lk} = \frac{1}{4\pi}\iint \delta\big(\mathbf{e}\cdot(\mathbf{e}_\sigma \times \mathbf{e}_s)\big)\; d\sigma\, ds = \frac{1}{4\pi}\; 3\iint \delta\mathbf{e}\cdot(\mathbf{e}_\sigma \times \mathbf{e}_s)\; d\sigma\, ds = 0,$$

where the last equality results from $\mathbf{e}_s$, $\mathbf{e}_\sigma$, and $\delta\mathbf{e}$ being coplanar (all are normal to $\mathbf{e}$).

## 3 The Hopf link




Let us first estimate the part of the link integral involving the part of $\mathbf{y}$ outside the radius $L$ sphere. We can say that the length of that part of the curve is equal to $kL$ for some constant $k$. Also there exists some constant $c$ such that $||\mathbf{y}-\mathbf{x}|| > c\,L$. We then compute

$$\begin{aligned} &\left|\int \frac{(\mathbf{y}-\mathbf{x})\cdot(\mathbf{y}'\times\mathbf{x}')}{||\mathbf{y}-\mathbf{x}||^3}\; d\sigma\right| \\ \le\; &\int \left|\frac{(\mathbf{y}-\mathbf{x})\cdot(\mathbf{y}'\times\mathbf{x}')}{||\mathbf{y}-\mathbf{x}||^3}\right|\; d\sigma \\ \le\; &\int \frac{||\mathbf{y}-\mathbf{x}||\,||\mathbf{y}'\times\mathbf{x}'||}{||\mathbf{y}-\mathbf{x}||^3}\; d\sigma \\ \le\; &\int \frac{1}{(c\,L)^2}\; d\sigma \\ =\; &\frac{k}{c^2\,L} \end{aligned}$$

which is arbitrarily small as $L \to \infty$.

For the part of the integral containing the straight line part of $\mathbf{y}(\sigma)$, we have the arc-length parameterisations of $\mathbf{x} = (\cos s \;\sin s \; 0)^T$ and $\mathbf{y} = (0\; 0\; \sigma)^T$ so that $\boldsymbol{y}'(\sigma) \times \boldsymbol{x}'(s) = (-\cos s \; -\sin s \; 0)^T$, $(\mathbf{y} - \mathbf{x}) \cdot (\mathbf{y}' \times \mathbf{x}') = 1$, and $||\mathbf{y} - \mathbf{x}|| = \sqrt{\sigma^2 + 1}$.

Hence,

$$\mathrm{Lk} = \frac{1}{4\pi} \int_0^{2\pi} \int_{-L}^{L} \frac{1}{(1+\sigma^2)^{\frac{3}{2}}} \, d\sigma \, ds + O(1/L) = \frac{2\pi}{4\pi} \int_{-L}^{L} \frac{1}{(1+\sigma^2)^{\frac{3}{2}}} \, d\sigma + O(1/L)$$

taking $\sigma = \sinh(z)$ so that $d\sigma = \cosh(z)\, dz$, we have that

$$\mathrm{Lk} = \frac{1}{2} \int_{\mathrm{arcsinh}(-L)}^{\mathrm{arcsinh}(L)} \frac{1}{\cosh^2(z)} \, dz + O(1/L) = \tanh(\mathrm{arcsinh}(L)) \to 1 \text{ as } L \to \infty$$

Finally, since the link integral does not change under deformations where $\mathbf{x}$ intersects itself, the second link can be deformed to a hopf link with the $\mathbf{x}$ curve covered twice. The domain of integration along $\mathbf{x}$ can therefore be split into two parts each of which are the same as the link integral of the Hopf link. Accordingly, we have $\mathrm{Lk} = 2$.