1 Unknotted curves and the Whitehead link

Let \( n \) be a unit vector normal to the plane separating \( x(s) \) and \( y(\sigma) \). For any real number \( \alpha \), define a closed curve \( Tr(\alpha; C_1) \) parametrised as \( \alpha n + x(s) \) which is just the curve \( C_1 \) translated. Since \( x(s) \) and \( y(\sigma) \) are closed and disjoint, there exist positive constants \( k_1, K_1 \) such that \( k_1 \leq ||x(s) - y(\sigma)|| \leq K_1 \) for any \( s, \sigma \). Also, upon parametrising \( x \) and \( y \) by arc-length, we have \( ||y'(\sigma) \times x'(s)|| \leq 1 \) since \( y'(\sigma) \) and \( x'(s) \) are both of norm 1.

Then,

\[
\frac{|(y(\sigma) - x(s) - \alpha n) \cdot (y'(\sigma) \times x'(s))|}{||y(\sigma) - x(s) - \alpha n||^3} = \frac{1}{\alpha^2} \frac{|(\frac{1}{\alpha} (y-x) - n) \cdot (y' \times x')|}{||\frac{1}{\alpha} (y-x) - n||^3} \leq \frac{1}{\alpha^2} \frac{\frac{1}{\alpha} ||y-x|| + ||n|| ||y' \times x'||}{||\frac{1}{\alpha} ||y-x|| - ||n||^3} \leq \frac{1}{\alpha^2} \frac{\frac{1}{\alpha} K_1 + 1}{||\frac{1}{\alpha} k_1 - 1||^3}
\]

and as such we can compute

\[
|Lk(Tr(\alpha; C_1), C_2)| = \frac{1}{4\pi} \left| \int_0^{t_1} \int_0^{t_2} \frac{(y(\sigma) - x(s) - \alpha n) \cdot (y'(\sigma) \times x'(s))}{||y(\sigma) - x(s) - \alpha n||^3} d\sigma ds \right| \leq \frac{1}{4\pi} \int_0^{t_1} \int_0^{t_2} \frac{\frac{1}{\alpha} K_1 + ||n||}{||\frac{1}{\alpha} ||y-x|| - ||n||^3} \frac{1}{\alpha} |l_1 l_2|
\]

At the same time, by exercise 2, we have that \( Lk(Tr(\alpha; C_1), C_2) = Lk(C_1, C_2) \) for any \( \alpha \) which means that we have that

\[
|Lk(C_1, C_2)| = |Lk(Tr(\alpha; C_1), C_2)| \leq \frac{1}{4\pi} \alpha^2 \frac{\frac{1}{\alpha} K_1 + 1}{||\frac{1}{\alpha} k_1 - 1||^3} l_1 l_2
\]

and thus, since that expression tends to 0 when \( \alpha \) tends to infinity, \( Lk(C_1, C_2) = 0 \).

Now, we also want to compute the linking number of the Whitehead link using the method of counting signed crossings. The signs of the various crossings can be seen to be
Accordingly, it is impossible to make a choice for \( \varphi \) planes at \((s, \sigma)\) and thus, since \( \lim_{r \to 0} \) has been computed that \( Lk = 0 \).

Another way to conclude that the link integral vanishes is to notice that if we reverse the orientation of one of \( C_1 \) or \( C_2 \) we get the original turned around a half turn (which is a transformation which can be accomplished by homotopy). As such, \( Lk = -Lk \) and the conclusion follows.

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Solution 1.  

\[
\mathbf{r}(s_0 + \epsilon, \sigma_0 + \nu) = \mathbf{r}(s_0, \sigma_0) + \epsilon \mathbf{r}_s(s_0, \sigma_0) + \nu \mathbf{r}_\sigma(s_0, \sigma_0) + \frac{\epsilon^2}{2} \mathbf{r}_{ss}(s_0, \sigma_0) + \epsilon \nu \mathbf{r}_{s\sigma}(s_0, \sigma_0) + \frac{\nu^2}{2} \mathbf{r}_{\sigma\sigma}(s_0, \sigma_0) + O(\epsilon^3) + O(\nu^3) + O(\epsilon \nu^2) + O(\nu \epsilon^2),
\]

\[
= \mathbf{r} - \epsilon \mathbf{t}^{[x]} + \nu \mathbf{t}^{[y]} - \frac{\epsilon^2}{2} \kappa^{[x]} \mathbf{n}^{[x]} + \frac{\nu^2}{2} \kappa^{[y]} \mathbf{n}^{[y]} + O(\epsilon^3) + O(\nu^3) + O(\epsilon \nu^2) + O(\nu \epsilon^2),
\]

where \( \kappa^{[x]} \) and \( \kappa^{[y]} \) are the normal curvature at \((s_0, \sigma_0)\).

Another way to conclude that the link integral vanishes is to notice that if we reverse the orientation of one of \( C_1 \) or \( C_2 \) we get the original turned around a half turn (which is a transformation which can be accomplished by homotopy). As such, \( Lk = -Lk \) and the conclusion follows.

Solution 2.  

In general, we have

\[
\mathbf{r}_s \wedge \mathbf{r}_\sigma = -x'(s) \wedge y'(\sigma)
= -\left(t^{[x]} + \epsilon \kappa^{[x]} \mathbf{n}^{[x]} + O(\epsilon^2)\right) \wedge \left(t^{[y]} + \nu \kappa^{[y]} \mathbf{n}^{[y]} + O(\nu^2)\right)
= t^{[y]} \wedge t^{[x]} + \epsilon \kappa^{[x]} t^{[y]} \wedge \mathbf{n}^{[x]} + \nu \kappa^{[y]} \mathbf{n}^{[y]} \wedge t^{[x]} + \text{h.o.t.},
\]

where \( \text{h.o.t.} \) stands for higher order terms.

At a pair of points \((s_0, \sigma_0)\) where the tangents align, the first term in (1) vanishes and accordingly, the vector \( \mathbf{n} \) cannot be defined as in the exercise sheet. To show that no definition of \( \mathbf{n} \) at \((s_0, \sigma_0)\) will provide a smooth function \( \mathbf{n} \) in an open neighbourhood of \((s_0, \sigma_0)\), we show that \( \lim_{(s_0, \sigma_0) \to (s_0, \sigma_0)} \mathbf{n}(s, \sigma) \) depends on the path of approach. Locally, we specify this path by choosing a number \( \phi \in [0, \pi) \setminus \{\pi/2\} \) and taking \( \nu = \frac{\kappa^{[x]}}{\kappa^{[y]}} \tan \phi \). Substituting this for \( \nu \) in (1) yields

\[
\mathbf{r}_s \wedge \mathbf{r}_\sigma = \epsilon \kappa^{[x]} \left(t^{[y]} \wedge \mathbf{n}^{[x]} + \tan \phi \mathbf{n}^{[y]} \wedge t^{[x]}\right) + \text{h.o.t.},
= \epsilon \kappa^{[x]} \left(t^{[y]} \wedge (\mathbf{n}^{[x]} \pm \tan \phi \mathbf{n}^{[y]})\right) + \text{h.o.t.},
\]

So we must distinguish between two cases: either \( \mathbf{n}^{[x]} = \pm \mathbf{n}^{[y]} \) or not. In the first (rare) case, we find \( \mathbf{r}_s \wedge \mathbf{r}_\sigma = \pm \epsilon \kappa^{[x]} (1 \pm \tan \phi) \mathbf{b}_y \) where the two \( \pm \) lead to four possibilities depending on the geometry of the curves but not on the path of approach. In this case, \( \mathbf{n} \) can actually be regularised by picking \( \mathbf{n}(s_0, \sigma_0) = \pm \mathbf{b}_y \).

In the second case, by far the most common, the two curves do not have parallel osculating planes at \((s_0, \sigma_0)\): \( \mathbf{n}^{[x]} \wedge \mathbf{n}^{[y]} \neq 0 \). In that case the direction of the vector \( \mathbf{r}_s \wedge \mathbf{r}_\sigma \) depends on \( \phi \). Accordingly, it is impossible to make a choice for \( \mathbf{n}(s_0, \sigma_0) \) that would lead to a smooth limit for all \( \phi \).
Solution 3. Ad absurdum Let \( N = r_s \wedge r_\sigma \). And assume there exist a positive number \( a > 0 \) and a regular curve
\[
\gamma : t \in (-a, a) \rightarrow (s(t), \sigma(t)),
\]
such that both \( \gamma(0) = (s_0, \sigma_0) \) and \( N(\gamma(t)) = 0 \) for all \( t \in (-a, a) \). Then compute
\[
N(\gamma(t)) = 0 \Rightarrow \frac{\partial N}{\partial s} \frac{ds}{dt} + \frac{\partial N}{\partial \sigma} \frac{d\sigma}{dt} = 0,
\]
\[
\Rightarrow (y_\sigma \wedge x_{ss}) \dot{s} + (y_{\sigma\sigma} \wedge x_s) \dot{\sigma} = 0.
\]
A scalar product of (3) by \( y_{\sigma\sigma} \) yields
\[
\dot{s} = 0 \quad \text{or} \quad (x_{ss} \wedge y_{\sigma\sigma}) \cdot y_\sigma = 0. \tag{4}
\]
But since the curves \( x \) and \( y \) are regular, we have \( x_s \neq 0 \) and \( y_\sigma \neq 0 \). Now \( N(\gamma) = 0 \) implies that every point in the image of \( \gamma \) respects the hypothesis of question 2: the tangent to \( x \) is parallel to that of \( y \) at \( \gamma(t) \). However, since \( x_s \) and \( y_\sigma \) are unit vectors, we know that \( x_{ss} \) (resp. \( y_{\sigma\sigma} \)) is perpendicular to \( x_s \) (resp. \( y_\sigma \)). But since \( x_s = \pm y_\sigma \), we also have that \( x_{ss} \) is perpendicular to \( y_\sigma \). Accordingly, the RHS proposition in the alternative (4) can not occur at \( (s_0, \sigma_0) \) since \( x_{ss} \wedge y_{\sigma\sigma} \) both cannot vanish and must be parallel to \( y_\sigma \). We conclude that the left hand side must occur:
\[
\dot{s}(\gamma(0)) = 0. \tag{5}
\]
Similarly, the scalar product of (3) with \( x_{ss} \) implies that
\[
\dot{\sigma}(\gamma(0)) = 0. \tag{6}
\]
Together (5,6) imply that \( \gamma(0) = 0 \) so that the curve \( \gamma \) can not be regular: a contradiction.

Solution 4. Assume that \( f \) vanishes for some \( (s_0, \sigma_0) \) (not necessarily meeting the hypothesis of question 2). The implicit function theorem guarantees that if
\[
f_s(s_0, \sigma_0) \neq 0, \quad \text{or} \quad f_\sigma(s_0, \sigma_0) \neq 0, \tag{7}
\]
then there exists a unique open curve \( \gamma \) through \( (s_0, \sigma_0) \) such that \( f(\gamma(t)) = 0 \).

To conclude we prove that (7) is true. Ad absurdum, if (7) is false then at \( (s_0, \sigma_0) \), we have
\[
\begin{cases}
f = 0, & \quad (y - x) \cdot (y_\sigma \wedge x_s) = 0, \\
f_s = 0, & \quad (y - x) \cdot (y_\sigma \wedge x_{ss}) = 0, \quad \iff \quad (y - x), y_\sigma, y_{\sigma\sigma}, x_s \text{ and } x_{ss} \text{ are coplanar}. \\
f_\sigma = 0, & \quad (y - x) \cdot (y_{\sigma\sigma} \wedge x_s) = 0.
\end{cases} \tag{8}
\]
In particular, the last statement implies that \( x \) and \( y \) share an osculating plane at \( (s_0, \sigma_0) \): a contradiction.

Solution 5. If \( (s_0, \sigma_0) \) is such that \( x_s \) is parallel to \( y_\sigma \), then \( (r_s \wedge r_\sigma)|_{(s_0, \sigma_0)} = 0 \) and in particular \( f(s_0, \sigma_0) = 0 \). Then question 4 implies that such a curve exists.