Differential Geometry of Framed Curves	Version 1.1
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Session 6: Solutions	T. Lessinnes.

1 Curves on a sphere

1.1 A general curve lying on a shpere

1. For a curve on a sphere, $\mathbf{w} = R\,\mathbf{N}$, so we have $\mathbf{w}' = R'\,\mathbf{N} + R\,\mathbf{N}'$. But $\mathbf{w}' = \mathbf{v}$ and $R' = 0$ which yields the result: $\mathbf{N}' = \frac{\mathbf{v}}{R}$.

2. In general, we have for the Darboux vector $\mathbf{u} = \mathsf{u}_1\mathbf{N} \wedge \mathbf{v} + \mathsf{u}_2\mathbf{N} + \mathsf{u}_3\mathbf{v}$. Then

$$\begin{aligned}
\mathbf{N}' &= \mathbf{u} \wedge \mathbf{N} \\
&= (\mathsf{u}_1\mathbf{N} \wedge \mathbf{v} + \mathsf{u}_2\mathbf{N} + \mathsf{u}_3\mathbf{v}) \wedge \mathbf{N} \\
&= \mathsf{u}_1\mathbf{v} - \mathsf{u}_3\mathbf{N} \wedge \mathbf{v} \\
&= \frac{\mathbf{v}}{R},
\end{aligned}$$

   where the last equality comes from the result of the previous question.

   The two last lines imply that $\mathsf{u}_1 = \frac{1}{R}$ and $\mathsf{u}_3 = 0$. Thus the general form of $\mathbf{u}$ is

$$\mathbf{u}(\sigma) = \frac{1}{R}\,(\mathbf{N}(\sigma) \wedge \mathbf{v}(\sigma)) + \mathsf{u}_2(\sigma)\,\mathbf{N}(\sigma) \tag{1}$$

   for some function $\mathsf{u}_2(\sigma)$.

   That is the twist $\mathsf{u}_3(\sigma)$ of the sphere framing vanishes. The normal curvature of the sphere $\mathsf{u}_1(\sigma)$ is fixed as $1/R$ (for this special case of a sphere, it is independent both of position and of direction $\mathbf{v}$. And finally, the geodesic curvature u_2 is arbitrary (i.e.: can change from curve to curve).

3. As we have $\kappa = ||\mathbf{v}'|| = ||\mathbf{u} \wedge \mathbf{v}||$, we use the result (1) in the previous exercise to get

$$\kappa = \sqrt{\mathsf{u}_2^2 + \frac{1}{R^2}}.$$

4. We have

$$\mathbf{u}_F = \tau\,\mathbf{v} + \kappa\,\mathbf{b}, \tag{2}$$

   where $\mathbf{u}_F$ is the Darboux vector of the Frenet frame $\mathbf{b}$ and is the binormal to $\mathbf{w}$. Because D and the Frenet frame are both adapted, they are connected by the following rotation

$$\begin{pmatrix}\mathbf{n} & \mathbf{b} & \mathbf{v}\end{pmatrix} = \begin{pmatrix}\mathbf{N} \wedge \mathbf{v} & \mathbf{N} & \mathbf{v}\end{pmatrix} \begin{pmatrix}\cos\phi & -\sin\phi & 0 \\ \sin\phi & \cos\phi & 0 \\ 0 & 0 & 1\end{pmatrix} \tag{3}$$

   for some function $\phi(\sigma)$ (that represents the angle between $\mathbf{N} \wedge \mathbf{v}$ and the principal normal vector $\mathbf{n}$ to the curve $\mathbf{w}$).

   The relation between Darboux vectors of two adapted framings reads (exercise 1, serie 3)

$$\mathbf{u}_F = \mathbf{u} + \phi'\,\mathbf{v}. \tag{4}$$

Substituting (1,4) and (3) in respectively the left and right hand side of (2) gives

$$\cos\phi = \frac{\mathsf{u}_2}{\kappa}, \quad \sin\phi = -\frac{1}{R\kappa}, \quad \text{and} \quad \phi' = \tau.$$

Next, differentiate $\cot\phi = -\mathsf{u}_2 R$ to obtain $(\cot\phi)' = -\frac{\phi'}{\sin^2\phi} = -\mathsf{u}_2' R$. Finally, we have

$$\tau = \phi' = \mathsf{u}_2' \, R \, \sin^2\phi = \frac{\mathsf{u}_2' R}{\kappa^2 R^2} = \frac{\mathsf{u}_2' \, R}{1 + R^2 \, \mathsf{u}_2^2}.$$

Alternative solution. Since $\mathbf{w}$ is required to lie on the surface of a sphere, we have, from question 8 on series 1

$$\frac{1}{\kappa^2} + \left(\frac{1}{\kappa}\right)^{\prime 2} \frac{1}{\tau^2} = R^2$$

So

$$\left(R^2 - \frac{1}{\kappa^2}\right)\tau^2 = \left(\frac{\kappa'}{\kappa^2}\right)^2$$

Rearranging further, we have

$$\tau^2 = \left(\frac{\kappa'}{\kappa^2}\right)^2 \frac{1}{R^2 - \frac{1}{\kappa^2}} = \frac{\kappa'^2}{(R^2\kappa^2 - 1)\kappa^2}$$

From part 3 of this question we can calculate

$$\kappa' = \frac{\mathsf{u}_2'\mathsf{u}_2}{\sqrt{\mathsf{u}_2 + \frac{1}{R^2}}} = \frac{\mathsf{u}_2'\mathsf{u}_2}{\kappa}$$

Therefore

$$\tau^2 = \frac{(\mathsf{u}_2'\mathsf{u}_2)^2}{\kappa^2}\frac{1}{(R^2\kappa^2 - 1)\kappa^2} = \frac{(\mathsf{u}_2'\mathsf{u}_2)^2}{(\mathsf{u}_2^2 + \frac{1}{R^2})}\frac{1}{(R^2\mathsf{u}^2)(\mathsf{u}_2^2 + \frac{1}{R^2})} = \frac{\mathsf{u}_2'^2}{(\mathsf{u}_2^2 + \frac{1}{R^2})^2 R^2} = \left(\frac{\mathsf{u}_2' R}{\mathsf{u}_2^2 R^2 + 1}\right)^2$$

Hence the result. The two choices of sign of the square root correspond to the two choices of the unit normal field $\mathbf{N}$.

5. Consider the following circle parameterised by arc-length

$$\mathbf{w}(s) = \begin{pmatrix} r \, \cos s/r \\ r \, \sin s/r \\ R\sqrt{1 - r^2/R^2} \end{pmatrix}. \tag{5}$$

Then compute

$$\mathbf{N}(s) = \begin{pmatrix} r/R \, \cos s/r \\ r/R \, \sin s/r \\ \sqrt{1 - r^2/R^2} \end{pmatrix}, \qquad \mathbf{v}(s) = \mathbf{w}'(s) = \begin{pmatrix} -\sin s/r \\ \cos s/r \\ 0 \end{pmatrix}, \tag{6}$$

$$\mathbf{v}'(s) = \begin{pmatrix} -1/r \, \cos s/r \\ -1/r - \, \sin s/r \\ 0 \end{pmatrix}, \qquad \mathbf{N} \times \mathbf{v} = \begin{pmatrix} -\cos s/r \, \sqrt{1 - r^2/R^2} \\ -\sin s/r \, \sqrt{1 - r^2/R^2} \\ r/R \end{pmatrix}. \tag{7}$$

For a general framing $\{\mathbf{d}_1, \mathbf{d}_2, \mathbf{d}_3\}$ and its Darboux vector $\mathbf{u}$, we have $\mathbf{d}_3' = \mathbf{u} \times \mathbf{d}_3 = \mathsf{u}_2 \, \mathbf{d}_1 - \mathsf{u}_1 \, \mathbf{d}_2$. Hence, we have $\mathsf{u}_2 = \mathbf{d}_3' \cdot \mathbf{d}_1$ and $\mathsf{u}_1 = -\mathbf{d}_3' \cdot \mathbf{d}_2$. In our case, this gives

$$\mathsf{u}_2 = \mathbf{v}' \cdot (\mathbf{N} \times \mathbf{v}) = \frac{1}{r}\sqrt{1 - \frac{r^2}{R^2}}, \qquad \text{and} \qquad \mathsf{u}_1 = -\mathbf{v}' \cdot \mathbf{N} = 1/R. \tag{8}$$

1.2 Tangent indicatrix

In this exercise we have a curve $\mathbf{x}(s)$ such that

$$\mathbf{x}'(s) = \mathbf{w}(\sigma(s)). \tag{9}$$

1. Defining $\mathbf{n}(s)$ and $\kappa(s)$ the principal normal and curvature of $\mathbf{x}(s)$, we have

$$\mathbf{x}''(s) = \kappa\,\mathbf{n} = \frac{d}{ds}\mathbf{w}(\sigma(s)) = \frac{d\mathbf{w}(\sigma)}{d\sigma}\frac{d\sigma}{ds} = \mathbf{v}\frac{d\sigma}{ds}. \tag{10}$$

   Hence $\kappa = \pm\frac{d\sigma}{ds}$. Then, since $\kappa(s) > 0$ for all s, we can always choose the orientation of the parameterisation σ of $\mathbf{w}$ so as to have $\kappa = \frac{d\sigma}{ds}$. In that case, Eq. (10) further implies that $\mathbf{n} = \mathbf{v}$.

2. Define

$$\begin{aligned}
\mathbf{t}(s) &:= \mathbf{x}'(s) \overset{(9)}{=} \mathbf{w}(\sigma(s)) = \mathbf{N}(\sigma(s)),\\
\mathbf{n}(s) &:= \frac{\mathbf{t}'(s)}{\kappa} \overset{(\mathbf{1.2.1})}{=} \mathbf{v}(s),\\
\mathbf{b}(s) &:= \mathbf{t}(s) \wedge \mathbf{n}(s) = \mathbf{N}(s) \wedge \mathbf{v}(s).
\end{aligned}$$

   Then the Frenet frame $F^{[\mathbf{x}]} = \begin{pmatrix}\mathbf{n} & \mathbf{b} & \mathbf{t}\end{pmatrix}$ of the curve $\mathbf{x}$ can be obtained from $D = \begin{pmatrix}\mathbf{N}\wedge\mathbf{v} & \mathbf{N} & \mathbf{v}\end{pmatrix}$ by the following change of basis:

$$\begin{aligned}
F^{[\mathbf{x}]} &= \begin{pmatrix}\mathbf{v} & \mathbf{N}\wedge\mathbf{v} & \mathbf{N}\end{pmatrix}\\
&= D\begin{pmatrix}0 & 1 & 0\\ 0 & 0 & 1\\ 1 & 0 & 0\end{pmatrix}.
\end{aligned}$$

3. Compute

$$\begin{aligned}
\frac{d}{ds}\mathbf{b} &= \frac{d}{ds}(\mathbf{N}\wedge\mathbf{v})\\
&= \frac{d}{d\sigma}(\mathbf{N}\wedge\mathbf{v})\frac{d\sigma}{ds}\\
&= \mathbf{u}\wedge(\mathbf{N}\wedge\mathbf{v})K\\
&= K\left(\frac{1}{R}\mathbf{N}\wedge\mathbf{v} + \mathsf{u}_2\mathbf{N}\right)\wedge(\mathbf{N}\wedge\mathbf{v})\\
&= -K\mathsf{u}_2\mathbf{v}.
\end{aligned}$$

   Comparing this result with the Frenet formula $\mathbf{b}' = -T\mathbf{n}$, where T is the torsion of the curve $\mathbf{x}$ yields the result.

2 Generalised Helices

The tantrix of $\mathbf{x}$ is an arc of a circle if and only if there is a constant unit vector $\mathbf{c}$ such that $\mathbf{t}\cdot\mathbf{c}$ is constant. We differentiate to get

$$K\mathbf{n}\cdot\mathbf{c} = 0$$

Since the curvature K never vanishes (by assumption) we must have $\mathbf{n}\cdot\mathbf{c} = 0$ so

$$\mathbf{c} = \sin\alpha\mathbf{t} + \cos\alpha\mathbf{b}$$

where α is constant since $\mathbf{t} \cdot \mathbf{c}$ is constant. Differentiating yields

$$\mathbf{0} = \sin\alpha \mathbf{t}' + \cos\alpha \mathbf{b}' = (K \sin\alpha - T\cos\alpha)\mathbf{n}$$

Since $\mathbf{n} \neq \mathbf{0}$ we must have

$$\frac{T}{K} = \tan\alpha$$

These arguments apply in reverse to show the opposite implication.

Note that the special case of $T \equiv 0$ means that any planar curve is a generalised helix.

3 Tangent indicatrix of a closed curve

Solution 2.1. First we show that if we can find a strictly positive function $f(\sigma)$ such that

$$\exists f(\sigma) > 0 \ \forall\sigma : \int_a^b f(\sigma)\mathbf{w}(\sigma)d\sigma = \mathbf{0} \Rightarrow \mathbf{w} \text{ not enclosed in a hemisphere.} \tag{11}$$

Ad absurdum, suppose $\mathbf{w}$ is in a hemisphere. Then there exists $\mathbf{e}$ a unit vector perpendicular to the great circle limiting the hemisphere such that $\mathbf{w}(\sigma) \cdot \mathbf{e} > 0$ for all σ. Then

$$\int_a^b f(\sigma)(\mathbf{w}(\sigma) \cdot \mathbf{e})\, d\sigma > 0, \tag{12}$$

contradicts the LHS of (11).

Next we prove that

$$\mathbf{w} \text{ not enclosed in a hemisphere} \Rightarrow \exists f > 0 : \int_a^b f(\sigma)\mathbf{w}(\sigma)d\sigma = \mathbf{0}. \tag{13}$$

Consider the set $K = \{\int_a^b f(\sigma)\mathbf{w}(\sigma)\, d\sigma \mid f > 0\} \subset \mathbb{R}^3$. The set K is convex, that is $\forall \mathbf{p}, \mathbf{q} \in K$, $\forall \lambda \in [0,1] : \lambda\mathbf{p} + (1-\lambda)\mathbf{q} \in K$. Indeed if $\mathbf{p} \in K$ and $\mathbf{q} \in K$, then by definition of K, there exist strictly positive functions $p(\sigma)$ and $q(\sigma)$ such that $\mathbf{p} = \int_a^b p(\sigma)\mathbf{w}(\sigma)\, d\sigma$ and $\mathbf{q} = \int_a^b q(\sigma)\mathbf{w}(\sigma)\, d\sigma$. Then for any $\lambda \in [0,1]$, we have $\lambda\, \mathbf{p} + (1-\lambda)\mathbf{q} = \int_a^b \big(\lambda q(\sigma) + (1-\lambda)p(\sigma)\big)\mathbf{w}(\sigma)\, d\sigma \in K$ since $\lambda q(\sigma) + (1-\lambda)p(\sigma)$ is a strictly positive function. Also note that it easy to build a sequence of strictly positive functions $f_i(\sigma, \sigma^\star)$ such that $\lim_{i\to\infty} \int_a^b f_i(\sigma,\sigma^\star)\mathbf{w}(\sigma)d\sigma = \mathbf{w}(\sigma^\star)$. Accordingly, the image of $\mathbf{w}$ is in the closure of K. Finally, the set $O = \{\mathbf{0}\}$ consisting of only the origin is also convex.

Then *ad absurdum*, if there is no $f > 0$ such that $\int_a^b f(\sigma)\mathbf{w}(\sigma)d\sigma = \mathbf{0}$, then the convex sets O and K are disjoint, the Minkowski separation theorem applies and there exists a vector $\mathbf{v}$ and a constant c such that the plane $H = \{\mathbf{a} \in \mathbb{R}^3 : \mathbf{v} \cdot \mathbf{a} = c\}$ separates O and K. A fortiori, K and its closure are contained on one side of the plane $H_0 = \{\mathbf{a} \in \mathbb{R}^3 : \mathbf{v} \cdot \mathbf{a} = 0\}$ so that the image of $\mathbf{w}$ is contained in the hemisphere the border of which is included in H_0.

Solution 2.2. If $\mathbf{w}$ is a tantrix of a closed curve $\mathbf{x}(s)$ parameterised by arc-length and where $s \in [0, \ell]$, then we have

$$0 = \mathbf{x}(\ell) - \mathbf{x}(0) = \int_0^\ell \mathbf{x}'(s)ds = \int_a^b \frac{\mathbf{w}(\sigma)}{\frac{d\sigma}{ds}} d\sigma \tag{14}$$

where σ is arc-length along $\mathbf{w}$. Apply the result of 2.1 with the function $f = \left(\frac{d\sigma}{ds}\right)^{-1}$ to conclude.

If $\mathbf{w}$ is not included in a hemisphere, then there exists a functions f such that $\int_a^b f(\sigma)\mathbf{w}(\sigma)d\sigma = 0$. Define the function $s(\sigma)$ that solve the IVP $s'(\sigma) = f(\sigma)$ with initial value $s(a) = 0$. Invert the monotonically increasing function $s(\sigma)$ to obtain $\sigma(s)$. Then the solution of $\mathbf{x}'(s) = \mathbf{w}(\sigma(s))$ with initial value $\mathbf{x}(0) = \mathbf{0}$ is a closed curve the tantrix of which is $\mathbf{w}$.

Finally, yes it is possible to build two closed curves with non-intersecting tantrices. For instances pick the two borders of the white line across a tennis ball. They have the centre of the ball in their convex hull. Hence it is possible to build closed curves which admit those two non-intersecting lines on the sphere as tantrices.