Differential Geometry of Framed Curves
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Session 7: Solutions — A. Flynn

Solution 1 Let $f(t,\xi) = \mathbf{d}(t,\xi)\cdot\mathbf{d}(t,\xi)$. At a minimum (t_0,ξ_0) of f we have

$$\frac{\mathrm{d}f}{\mathrm{d}t} = 2\mathbf{x}'(t_0)\cdot\mathbf{d}(t_0,\xi_0) = 0 \quad \text{and} \quad \frac{\mathrm{d}f}{\mathrm{d}\xi} = 2\mathbf{y}'(\xi_0)\cdot\mathbf{d}(t_0,\xi_0) = 0$$

Therefore $\mathbf{d}(t_0,\xi_0)$ is mutually orthogonal to $\mathbf{x}'(t_0)$ and $\mathbf{y}'(\xi_0)$.

Solution 2 Both factors change sign in the numerator while the denominator is left unchanged.

Solution 3

- Under a translation $\mathbf{x}\mapsto\mathbf{x}+\mathbf{a}$ for some $\mathbf{a}\in\mathbb{R}^3$:

$$(\mathbf{x}(\sigma)+\mathbf{a}) - (\mathbf{x}(s)+\mathbf{a}) = \mathbf{x}(\sigma)-\mathbf{x}(s)$$

$$(\mathbf{x}(\sigma)+\mathbf{a})'\wedge(\mathbf{x}(s)+\mathbf{a})' = \mathbf{x}'(\sigma)\wedge\mathbf{x}'(s)$$

- Under a rotation $\mathbf{x}\mapsto R\mathbf{x}$ for some $R\in\mathrm{SO}(3)$:

$$R\mathbf{x}(\sigma) - R\mathbf{x}(s) = R(\mathbf{x}(\sigma)-\mathbf{x}(s)).$$

$$\begin{aligned}(R\mathbf{x}(\sigma))'\wedge(R\mathbf{x}(s))' &= (R\mathbf{x}'(\sigma))^\times(R\mathbf{x}'(s)) \\ &= |R|R^{-T}(\mathbf{x}'(\sigma))^\times R^{-1}(R\mathbf{x}'(s)) \\ &= R\,(\mathbf{x}'(\sigma)\wedge\mathbf{x}'(s)). \qquad (1)\end{aligned}$$

  Accordingly, under rigid rotation, the denominator is left unchanged while the numerator becomes

$$\begin{aligned}(R\,\mathbf{x}(\sigma) - R\,\mathbf{x}(s))\cdot(R\mathbf{x}(\sigma))'\wedge(R\mathbf{x}(s))' &= \Big[R\,(\mathbf{x}(\sigma)-\mathbf{x}(s))\Big]\cdot\Big[R\,(\mathbf{x}'(\sigma)\wedge\mathbf{x}'(s))\Big], \\ &= (\mathbf{x}(\sigma)-\mathbf{x}(s))\cdot R^T\,R\,(\mathbf{x}'(\sigma)\wedge\mathbf{x}'(s)), \\ &= (\mathbf{x}(\sigma)-\mathbf{x}(s))\cdot(\mathbf{x}'(\sigma)\wedge\mathbf{x}'(s)). \qquad (2)\end{aligned}$$

- Under a dilation $\mathbf{x}\mapsto\lambda\mathbf{x}$, we get the factor λ^3 both in the nominator and the denominator, so Wr is invariant under dilations as well.

Note that if $R\in O(3)\setminus SO(3)$, then $|R|=-1$ and a minus sign would appear in (1). Accordingly, the Writhe flips sign under isometries of determinant -1.

Solution 4 For a planar curve, the three vectors $(\mathbf{x}(\sigma)-\mathbf{x}(s))$, $\mathbf{x}'(\sigma)$ and $\mathbf{x}'(s)$ are coplanar. Their scalar triple product vanishes and $I_{\mathrm{Wr}}=0$.

We can obtain the same result making use of question 2 by noticing that the curve is left unchanged by reflection through its plane (an isometry of determinant -1) while the Writhe must change sign.

Solution 5 Given $\mathbf{e}\cdot(\mathbf{e}_s\wedge\mathbf{e}_\sigma)=0$, we have

$$(\mathbf{x}(s)-\mathbf{x}(\sigma))\cdot(\mathbf{x}'(s)\wedge\mathbf{x}'(\sigma))=0,$$

which is equivalent to

$$\mathbf{x}'(s)\cdot[\mathbf{x}'(\sigma)\wedge(\mathbf{x}(s)-\mathbf{x}(\sigma))]=0 \tag{3}$$

Differentiate (3) with respect to s to get

$$\mathbf{x}''(s)\cdot[\mathbf{x}'(\sigma)\wedge(\mathbf{x}(s)-\mathbf{x}(\sigma))]=0. \tag{4}$$

But $\mathbf{x}'=v\mathbf{t}$ with $v=||\mathbf{x}'||=\frac{\mathrm{d}S}{\mathrm{d}s}$ where S is the arc-length along $\mathbf{x}$, so we have

$$\mathbf{x}''=v'\mathbf{t}+v\mathbf{t}'=v'\mathbf{t}+v^2\kappa\mathbf{n}. \tag{5}$$

Eqs. (3-5) imply that $\mathbf{x}'(\sigma)\wedge(\mathbf{x}(s)-\mathbf{x}(\sigma))$ is perpendicular to the $(\mathbf{t},\mathbf{n})$-plane so it must be along $\mathbf{b}(s)$:

$$\mathbf{b}(s)=\pm\frac{\mathbf{x}'(\sigma)\wedge(\mathbf{x}(s)-\mathbf{x}(\sigma))}{||\mathbf{x}'(\sigma)\wedge(\mathbf{x}(s)-\mathbf{x}(\sigma))||}. \tag{6}$$

Differentiate (6) with respect to s to get

$$\mathbf{b}'(s)=\pm\frac{\mathbf{x}'(\sigma)\wedge\mathbf{x}'(s)}{||\mathbf{x}'(\sigma)\wedge(\mathbf{x}(s)-\mathbf{x}(\sigma))||}-\mathbf{w},$$

where $\mathbf{w}$ is parallel to $\mathbf{b}$.

Finally $\frac{d\mathbf{b}}{ds}=\frac{d\mathbf{b}}{dS}\frac{dS}{ds}=-v\tau\mathbf{n}$ so that

$$\begin{aligned}-v\,\kappa\,\tau=\mathbf{b}'(s)\cdot\kappa\mathbf{n}(s) &= \mathbf{b}'(s)\cdot\frac{\mathbf{x}''(s)-\frac{v'(s)}{v(s)}\mathbf{x}'(s)}{v^2}\\ &= \frac{\mathbf{x}'(\sigma)\wedge\mathbf{x}'(s)}{||\mathbf{x}'(\sigma)\wedge(\mathbf{x}(s)-\mathbf{x}(\sigma))||}\cdot\frac{\mathbf{x}''(s)}{v^2}\\ &= \frac{(\mathbf{x}'(s)\wedge\mathbf{x}''(s))\cdot\mathbf{x}'(\sigma)}{||\mathbf{x}'(\sigma)\wedge(\mathbf{x}(s)-\mathbf{x}(\sigma))||v^2}.\end{aligned}$$

But $(\mathbf{x}'(s)\wedge\mathbf{x}''(s))$ is perpendicular to both $\mathbf{t}$ and $\mathbf{n}$ and so it must be parallel to $\mathbf{x}'(\sigma)\wedge(\mathbf{x}(s)-\mathbf{x}(\sigma))$ and so it is perpendicular to $\mathbf{x}'(\sigma)$. Hence the torsion vanishes identically. The curve is planar.

Notice that we are in the special case mentioned in question 2 on sheet 6. Namely $\tau\equiv 0$.

Solution 6

Step 1: Consider the planar problem of finding a circle through two points A and B and tangent to a unit vector $\mathbf{t}$ at B. Let ℓ be the straight line through A and B, θ be the angle between ℓ and the line $q=\operatorname{span}\mathbf{t}$, a be the distance between A and B, O the centre of the circle, p the perpendicular to AB through O, and P and Q the respective intersections of p with AB and q. The angle BOP is θ since the triangles BOP and BQP are similar). Then in the right-angled triangle BOP, we find

$$\sin\theta=\frac{a}{2\,R}, \tag{7}$$

where R is the radius of the circle. See figure 1.

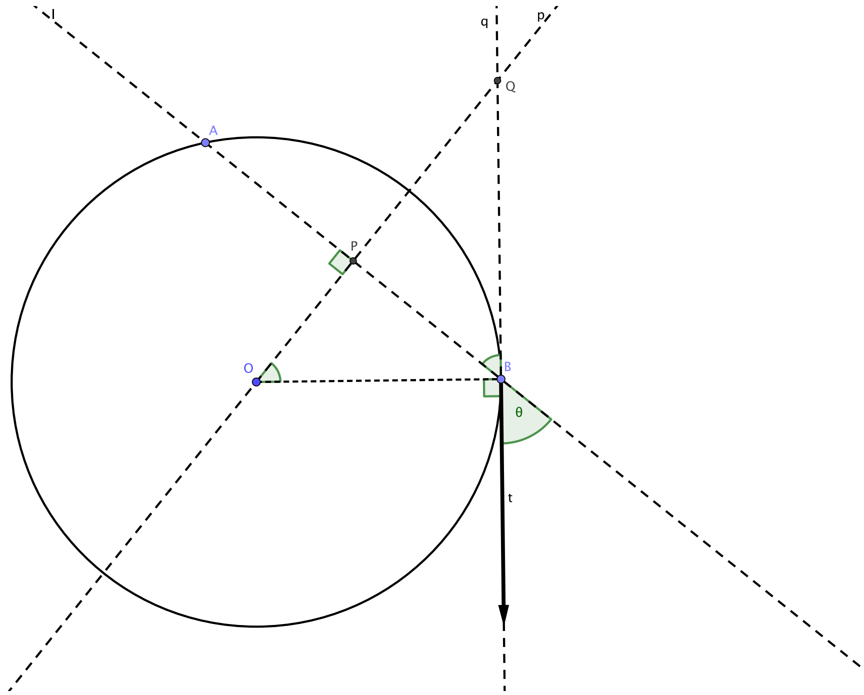

Figure 1: A circle in point contact at A and tangent contact at B.

Step 2: Modify the expression for writhe. We have already showed (technically, we did it for Lk but the proof is identical for Wr) that

$$Wr = \frac{1}{4\pi} \int\int \mathbf{e} \cdot (\mathbf{e}_s \times \mathbf{e}_\sigma)\ ds\, d\sigma, \tag{8}$$

where $\mathbf{e}(s,\sigma) = \frac{\mathbf{x}(\sigma)-\mathbf{x}(s)}{||\mathbf{x}(\sigma)-\mathbf{x}(s)||}$. We have

$$(\mathbf{e} \times \mathbf{e}_s) \times (\mathbf{e} \times \mathbf{e}_\sigma) = \Big(\mathbf{e} \cdot (\mathbf{e}_s \times \mathbf{e}_\sigma)\Big)\ \ \mathbf{e}, \tag{9}$$

Also

$$\mathbf{e} \times \mathbf{e}_\sigma = \mathbf{e} \times \frac{\mathbf{x}'(\sigma)}{||\mathbf{x}(\sigma) - \mathbf{x}(s)||} = \mathbf{e} \times \frac{\mathbf{x}'(\sigma)}{a(s,\sigma)} = \frac{\sin\theta_1}{a(s,\sigma)}\mathbf{n}_1 \overset{(7)}{=} \frac{\mathbf{n}_1}{d(s,\,\sigma)}, \quad \text{and} \quad \mathbf{e} \times \mathbf{e}_s = \frac{\mathbf{n}_2}{d(\sigma,\,s)} \tag{10}$$

where $a(s,\,\sigma) = ||\mathbf{x}(\sigma) - \mathbf{x}(s)||$, $\mathbf{n}_1$ (resp. $\mathbf{n}_2$) is the unit vector along $\mathbf{e} \times \mathbf{e}_\sigma$ (resp. $\mathbf{e} \times \mathbf{e}_s$) and θ_1 (resp. θ_2) is the angle between $\mathbf{e}$ and $\mathbf{e}_\sigma$ (resp. $\mathbf{e}_s$). Substituting (10) in (9) gives

$$\frac{\sin\psi}{d(s,\sigma)d(\sigma,s)}\mathbf{e} = \Big(\mathbf{e} \cdot (\mathbf{e}_s \times \mathbf{e}_\sigma)\Big)\ \ \mathbf{e},$$

where ψ is the angle between $\mathbf{e} \times \mathbf{e}_\sigma$ and $\mathbf{e} \times \mathbf{e}_s$. Note that $\mathbf{e} \times \mathbf{e}_\sigma$ and $\mathbf{e} \times \mathbf{e}_s$ are the normal vectors to the planes spanned by $\{\mathbf{e}, \mathbf{x}'(\sigma)\}$ and $\{\mathbf{e}, \mathbf{x}'(s)\}$ respectively. Equivalently ψ is the angle between the planes themselves. See figure 2.

By using the signed angle convention introduced in lectures with $\mathbf{e}$ the vector mutually orthogonal to $\mathbf{e} \times \mathbf{e}_s$ and $\mathbf{e} \times \mathbf{e}_\sigma$ we can determine the sign of the angle ψ and remove the need to include the Sign$[\mathbf{e} \cdot (\mathbf{e}_s \times \mathbf{e}_\sigma)]$ term.

See [1] for more on global radius of curvature circles. See also figure 3 for an example of the writhe zodiacus of a curve and figure 4 for plots of the writhe integrand I_{Wr}.

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[1] Oscar Gonzalez and John H Maddocks *Global curvature, thickness, and the ideal shapes of knots* Proceedings of the National Academy of Sciences 4769–4773 (96) 1999

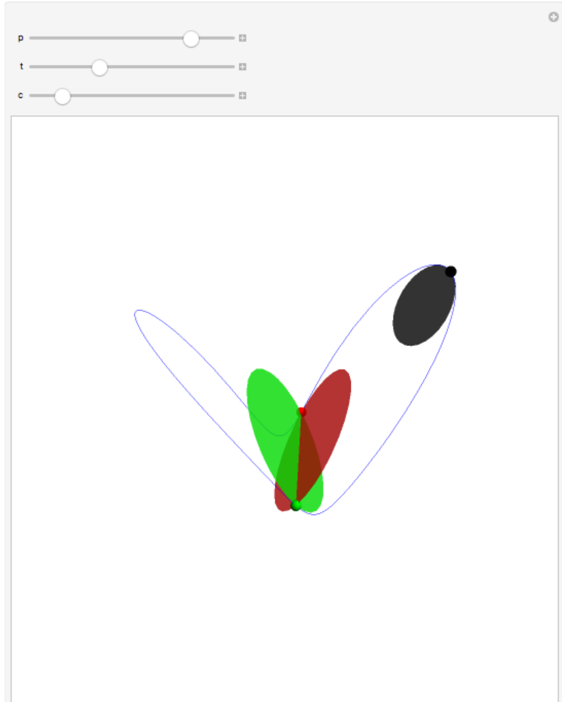

Figure 2: The red circle is in point contact with the curve (in blue) at the green point and tangent contact at the red point. The green circle is in point contact at the red point and tangent contact at the green point. The angle between the planes containing these two circles respectively is ψ. The black circle is the local radius of curvature circle at the black point.

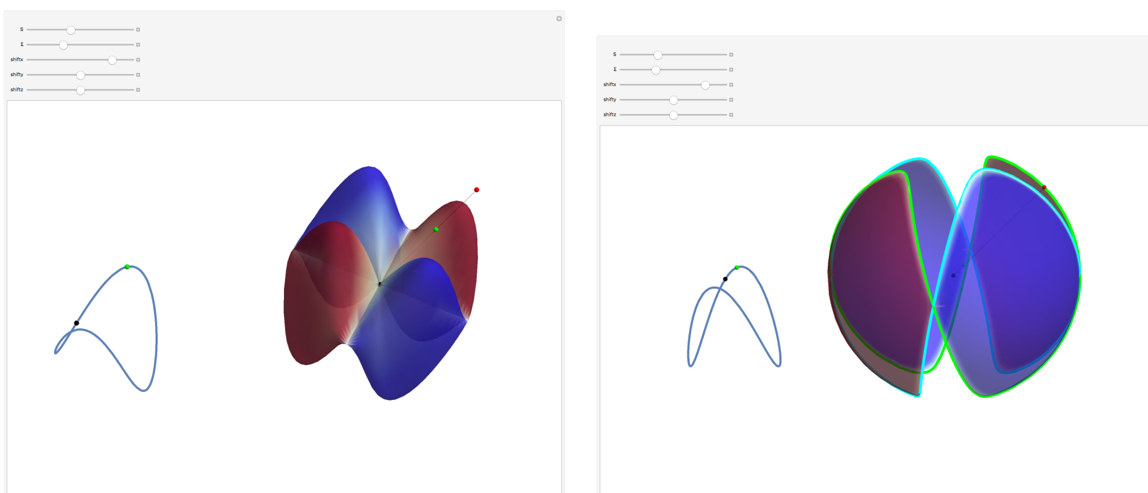

Figure 3: Left: The surface $\boldsymbol{\alpha}(s,\sigma) = \mathbf{x}(s) - \mathbf{x}(\sigma)$ for a closed curve (in blue). Notice that the surface cannot be regular, in particular whenever $s = \sigma$. Right: The projection $\mathbf{e}(s,\sigma)$ of $\boldsymbol{\alpha}$ onto the unit sphere. The cyan and green lines are the tantrices of $\mathbf{x}$ for the two possible choices of orientation. In both cases red indicates that the triple product $\mathbf{e}\cdot(\mathbf{e}_s \times \mathbf{e}_\sigma)$ is positive and blue indicates that the triple product is negative. Note that the zodiacus is single covered; the unit chord from the black point to the red point starts inside the surface and leaves the surface at the green point. Also note that the writhe of this curve is 0, this can be seen by observing that the curve has a plane of symmetry so, by question 3, $\mathrm{Wr}(\mathbf{x}) = -\mathrm{Wr}(\mathbf{x})$.

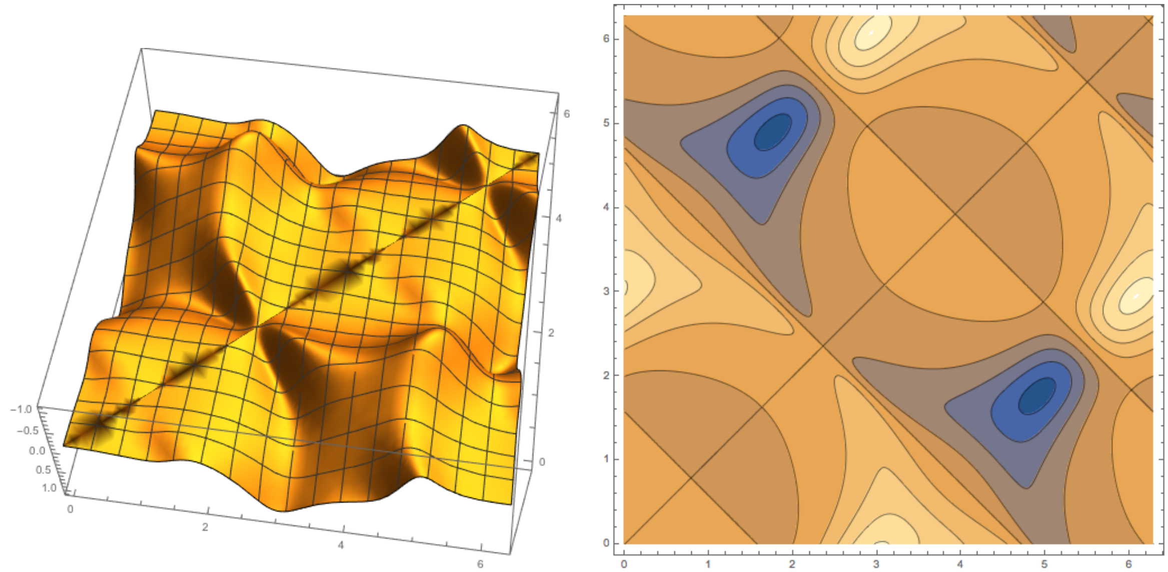

Figure 4: Surface (left) and contour (right) plot of the writhe integrand I_{Wr} for the curve in figure 2. The two long axes in the surface plot correspond to the two axes in the contour plot, they are s and σ respectively. The third axis in the surface plot is the value of $I_{\mathrm{Wr}}(s,\sigma)$. This value is represented by colour in the contour plot and level sets are drawn as lines.

Solution 7 Notice that we are using a non-standard parametrisation of a helix here so that r represents a dilation of the helix, thus the writhe is independent of r.

- $\mathbf{x}(s) = (r\cos s,\, r\sin s,\, rps), \quad \mathbf{x}'(s) = (-r\sin s,\, r\cos s,\, rp)$.
- $\mathbf{r}(s,\sigma) := \mathbf{x}(s) - \mathbf{x}(\sigma) = r(\cos s - \cos\sigma,\, \sin s - \sin\sigma,\, p(s-\sigma))$.
- $||\mathbf{r}|| = r\sqrt{4\sin^2\left(\frac{s-\sigma}{2}\right) + p^2(s-\sigma)^2}$.
- $$\begin{aligned}\mathbf{r}\cdot(\mathbf{r}_\sigma \wedge \mathbf{r}_s) &= r^3 \begin{vmatrix} \cos s - \cos\sigma & \sin s - \sin\sigma & p(s-\sigma) \\ -\sin s & \cos s & p \\ -\sin\sigma & \cos\sigma & p \end{vmatrix} \\ &= r^3 p\left[4\sin^2\left(\frac{s-\sigma}{2}\right) - (s-\sigma)\sin(s-\sigma)\right].\end{aligned}$$
- $$\begin{aligned}I_{\mathrm{Wr}} &= \frac{\mathbf{r}\cdot(\mathbf{r}_\sigma\wedge\mathbf{r}_s)}{||\mathbf{r}||^3} \\ &= \frac{p}{2}\cdot\frac{\sin^2\left(\frac{s-\sigma}{2}\right) - \frac{s-\sigma}{2}\sin\left(\frac{s-\sigma}{2}\right)\cos\left(\frac{s-\sigma}{2}\right)}{\left[\sin^2\left(\frac{s-\sigma}{2}\right) + p^2\left(\frac{s-\sigma}{2}\right)^2\right]^{\frac{3}{2}}}.\end{aligned}$$

As indicated in the hint we compute

$$\left(\frac{w}{\sqrt{\sin^2 w + p^2 w^2}}\right)' = \frac{\sin^2 w - w\sin w\cos w}{(\sin^2 w + p^2 w^2)^{\frac{3}{2}}}. \tag{11}$$

Finally, compute

$$
\begin{aligned}
\mathrm{Wr}(\mathbf{x}) &= \frac{1}{4\pi}\int_0^L d\sigma \int_0^L I_{\mathrm{Wr}}(s,\sigma)\,ds \\
&= \frac{1}{4\pi}\int_0^L d\sigma \int_{-\frac{\sigma}{2}}^{\frac{L-\sigma}{2}} \frac{p}{2}\,\frac{\sin^2 t - t\sin t\cos t}{(\sin^2 t + p^2 t^2)^{\frac{3}{2}}}\,2\,dt \quad \text{where } t=\frac{s-\sigma}{2} \\
&= \frac{p}{4\pi}\lim_{\varepsilon\to 0^+}\left\{\int_0^L d\sigma\left[\int_{-\frac{\sigma}{2}}^{-\varepsilon}\frac{\sin^2 t - t\sin t\cos t}{(\sin^2 t + p^2 t^2)^{\frac{3}{2}}}\,dt + \int_{\varepsilon}^{\frac{L-\sigma}{2}}\frac{\sin^2 t - t\sin t\cos t}{(\sin^2 t + p^2 t^2)^{\frac{3}{2}}}\,dt\right]\right\} \\
&\overset{(11)}{=} \frac{p}{4\pi}\int_0^L d\sigma \lim_{\varepsilon\to 0^+}\left\{\left[\frac{t}{\sqrt{\sin^2 t + p^2 t^2}}\right]_{-\frac{\sigma}{2}}^{-\varepsilon} + \left[\frac{t}{\sqrt{\sin^2 t + p^2 t^2}}\right]_{\varepsilon}^{\frac{L-\sigma}{2}}\right\}, \qquad (12) \\
&= \frac{p}{4\pi}\left[\int_0^L \frac{(L-\sigma)/2}{\sqrt{\sin^2(L-\sigma)/2 + p^2\,(L-\sigma)^2/4}}\,d\sigma + \int_0^L \frac{\sigma/2}{\sqrt{\sin^2\sigma/2 + p^2\,\sigma^2/4}}\,d\sigma\right] \\
&\qquad\qquad - \frac{p\,L}{2\pi}\lim_{\varepsilon\to 0}\frac{\varepsilon}{\sqrt{\sin^2\varepsilon + p^2\,\varepsilon^2}}, \\
&= \frac{p}{4\pi}\left[\int_{L/2}^0 \frac{-2\,u\,du/2}{\sqrt{\sin^2 u + p^2\,u^2}} + \int_0^{L/2}\frac{2\,v\,dv}{\sqrt{\sin^2 v + p^2\,v^2}} - \frac{2L}{\sqrt{1+p^2}}\right], \\
&\qquad\qquad \text{with } u=\frac{L-\sigma}{2} \text{ and } v=\frac{\sigma}{2}, \\
&= \frac{p}{\pi}\int_0^{\frac{L}{2}}\frac{v\,dv}{\sqrt{\sin^2 v + p^2 v^2}} - \frac{p\,L}{2\pi}\frac{1}{\sqrt{1+p^2}}. \qquad (13)
\end{aligned}
$$

To study the asymptotic behaviour, we first take a derivative w.r.t. L which yields

$$
\begin{aligned}
\frac{\mathrm{dWr}}{\mathrm{d}L} &= \frac{p}{2\pi}\left(\frac{L}{\sqrt{\sin^2 L + p^2 L^2}} - \frac{1}{\sqrt{1+p^2}}\right), \qquad (14) \\
&= \frac{p}{2\pi}\left(\frac{1}{\sqrt{\frac{1}{L^2}\sin^2 L + p^2}} - \frac{1}{\sqrt{1+p^2}}\right) \xrightarrow[L\to\infty]{} \frac{p}{2\pi}\left(\frac{1}{|p|} - \frac{1}{\sqrt{1+p^2}}\right). \qquad (15)
\end{aligned}
$$

Hence for large L, we have the following asymptotic behaviour for Wr:

$$
\mathrm{Wr}(L) \overset{L\to\infty}{\sim} \frac{p}{2\pi}\left(\frac{1}{|p|} - \frac{1}{\sqrt{1+p^2}}\right) L. \qquad (16)
$$

That is, the writhe is monotonic and has an asymptote with slope $\frac{p}{2\pi}\left(\frac{1}{|p|} - \frac{1}{\sqrt{1+p^2}}\right)$.

(Non-examinable from this point) This asymptotic result can also be derived somewhat more rigorously by the following analysis: change variables according to $g = 1/L$ so that $L \to \infty$ becomes $g \to 0^+$. The differential equation (14) becomes singular:

$$
-\,g^2\frac{\mathrm{dWr}}{\mathrm{d}g} = \frac{p}{2\pi}\left(\frac{1}{\sqrt{g^2\,\sin^2 1/g + p^2}} - \frac{1}{\sqrt{1+p^2}}\right). \qquad (17)
$$

We want to study the differential equation (17) as $g \to 0$. First note that $|p| \leq \sqrt{g^2 \sin^2 1/g + p^2} \leq \sqrt{g^2 + p^2}$. Accordingly

$$\frac{1}{|p|} \geq \frac{1}{\sqrt{g^2 \sin^2 1/g + p^2}} \geq \frac{1}{\sqrt{g^2 + p^2}} = \frac{1}{|p|} - \frac{g^2}{2\,|p|^3} + O(g^4), \tag{18}$$

so that

$$\frac{1}{\sqrt{g^2 \sin^2 1/g + p^2}} = \frac{1}{|p|} + O(g^2). \tag{19}$$

Next we expand $\mathrm{Wr}(g)$ close to $g = 0$ as

$$\mathrm{Wr}(g) = w_{-1}\, g^{-1} + w_0 + w_1\, g + O(g^2) \Rightarrow \mathrm{Wr}'(g) = -\frac{w_{-1}}{g^2} + w_1 + O(g), \tag{20}$$

where w_{-1}, w_0 and w_1 are constant coefficients that we will try to find.

Substituting (19) and (20) in (17) yields

$$-\, w_{-1} + w_1\, g^2 + O(g^3) = \frac{p}{2\pi} \left(\frac{1}{|p|} - \frac{1}{\sqrt{1+p^2}} + O(g^2) \right). \tag{21}$$

Hence we find $w_{-1} = \frac{p}{2\pi} \left(\frac{1}{|p|} - \frac{1}{\sqrt{1+p^2}} \right)$ but note that this analysis does not prescribe w_0 or w_1. Substituting our result and $g = L^{-1}$ in (20) gives

$$\mathrm{Wr}(L) = \frac{p}{2\pi} \left(\frac{1}{|p|} - \frac{1}{\sqrt{1+p^2}} \right) L + w_0 + w_1 \frac{1}{L} + O\left(\frac{1}{L^2} \right) \overset{L\to\infty}{\sim} \frac{p}{2\pi} \left(\frac{1}{|p|} - \frac{1}{\sqrt{1+p^2}} \right) L. \tag{22}$$

We see that we have gathered a little more information as to the behaviour of the solutions at higher order in $1/L$. To find w_0, we could write an expansion close to $L = 0$ and then enforce that both expansions must match smoothly at say $L = 1$. That sort of argument is useful to obtain approximate solutions to many non-linear problems and is well worth keeping in mind.