Differential Geometry of Framed Curves	VERSION 1
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SESSION 8	A. FLYNN

1 Yet another variant of the Euler-Rodrigues formula

For any $\mathbf{v} \in \mathbb{R}^3$, we have

$$\mathbf{n}^\times \mathbf{n}^\times \mathbf{v} = \mathbf{n}^\times (\mathbf{n}^\times \mathbf{v}) = \mathbf{n} \times (\mathbf{n} \times \mathbf{v}) = (\mathbf{n} \cdot \mathbf{v})\mathbf{n} - \mathbf{v} = (\mathbf{n} \otimes \mathbf{n} - \mathrm{Id})\, \mathbf{v}. \tag{1}$$

Since (1) holds for all $\mathbf{v}$ we have $\mathbf{n} \otimes \mathbf{n} = \mathrm{Id} + \mathbf{n}^\times \mathbf{n}^\times$ which can be substituted in Eq. (2) from the question sheet.

2 Eulers-Rodrigues parameters

(a) As discussed $q \in S^3$ is associated to a rotation of angle ϕ and axis $\mathbf{n}$. However, the rotation matrix $R(\phi, \mathbf{n}) \in SO(3)$ corresponding to a rotation of angle ϕ and axis along the unit vector $\mathbf{n}$ is given by the Rodrigues formula:

$$\begin{aligned} R(\phi, \mathbf{n}) &= \cos\phi\, \mathrm{Id} + (1 - \cos\phi)\, \mathbf{n} \otimes \mathbf{n} + \sin\phi\, \mathbf{n}^\times, \\ &= (\cos^2 \frac{\phi}{2} - \sin^2 \frac{\phi}{2})\, \mathrm{Id} + 2 \sin^2 \frac{\phi}{2}\, \mathbf{n} \otimes \mathbf{n} + 2 \cos \frac{\phi}{2} \sin \frac{\phi}{2}\, \mathbf{n}^\times, \\ &= (q_0^2 - \mathbf{q} \cdot \mathbf{q})\, \mathrm{Id} + 2\, \mathbf{q} \otimes \mathbf{q} + 2 q_0\, \mathbf{q}^\times = Q(q). \end{aligned} \tag{2}$$

Expanding (2) gives

$$\begin{aligned} Q(q) &= \begin{pmatrix} q_0^2 - q_1^2 - q_2^2 - q_3^2 & 0 & 0 \\ 0 & q_0^2 - q_1^2 - q_2^2 - q_3^2 & 0 \\ 0 & 0 & q_0^2 - q_1^2 - q_2^2 - q_3^2 \end{pmatrix} \\ &\quad + 2 \begin{pmatrix} q_1^2 & q_1 q_2 & q_1 q_3 \\ q_2 q_1 & q_2^2 & q_2 q_3 \\ q_3 q_1 & q_3 q_2 & q_3^2 \end{pmatrix} + \begin{pmatrix} 0 & -2\, q_0 q_3 & 2\, q_0 q_2 \\ 2\, q_0 q_3 & 0 & -2\, q_0 q_1 \\ -2\, q_0 q_2 & 2\, q_0 q_1 & 0 \end{pmatrix}. \end{aligned} \tag{3}$$

(b) Simply computing $Q(q)\mathbf{q}$ with $Q(q)$ as defined by (5) on the exercise sheet immediately shows $Q\mathbf{q} = \mathbf{q}$. However, note that it is even simpler to use the first equality in equation (4) of the exercise sheet. If $\mathbf{q} = 0$, the $q \in S^3$ implies $q_0 = \pm 1$ and $Q = \mathrm{Id}$.

(c) We have shown in question 2 of session 2 that

$$\mathrm{tr}(Q) = 1 + 2\cos\theta, \tag{4}$$

and also computing the Trace as the sum of the diagonal entries in (5) of the question sheet, we find

$$4q_4^2 - 1 = 1 + 2\cos\theta \Leftrightarrow q_4 = \pm\, \cos\frac{\theta}{2}. \tag{5}$$

(d) The fact that any $q \in S^3$ corresponds to a single $Q \in SO(3)$ is manifest from equation (5) of the exercise sheet.

Given a rotation matrix $R \in SO(3)$, define the vector $\mathbf{r} \in \mathbb{R}^3$ and the unit vector $\mathbf{n} \in S^2$ according to $\mathbf{r}^\times = R - R^T$ and $\mathbf{n} = \mathbf{r}/\|\mathbf{r}\|$. Also, define θ as the unique solution of $1 + 2\cos\theta = \mathrm{tr}\, R$ on $[0, \pi]$.

The angle θ and the vector $\mathbf{n}$ completely specify R. That is, if R_1 and R_2 have same θ and same $\mathbf{n}$, then $R_1 = R_2$. That is because for any vector $\mathbf{x}$, we have $\mathbf{x} = (\mathbf{x}\cdot\mathbf{n})\mathbf{n} + (\mathbf{n}\times\mathbf{x})\times\mathbf{n}$. The vector $(\mathbf{n}\times\mathbf{x})\times\mathbf{n}$ is perpendicular to $\mathbf{n}$ and we showed in question 3.c of session 2 that if $\mathbf{v}\perp\mathbf{w}$, for $\mathbf{w}$ along the axis of $Q\in SO(3)$, then $\mathbf{v} = \cos\theta\,\mathbf{v} + \sin\theta\,\mathbf{n}\times\mathbf{v}$ (in fact we almost did... make sure that you can close the argument). So we have

$$\begin{aligned} R\mathbf{x} &= R\Big[(\mathbf{x}\cdot\mathbf{n})\mathbf{n} + (\mathbf{n}\times\mathbf{x})\times\mathbf{n}\Big], \\ &= (\mathbf{x}\cdot\mathbf{n})\,R\,\mathbf{n} + R[(\mathbf{n}\times\mathbf{x})\times\mathbf{n}], \\ &= (\mathbf{x}\cdot\mathbf{n})\mathbf{n} + \cos\theta\,(\mathbf{n}\times\mathbf{x})\times\mathbf{n} + \sin\theta\,\mathbf{n}\times((\mathbf{n}\times\mathbf{x})\times\mathbf{n}), \\ &= (\mathbf{x}\cdot\mathbf{n})\mathbf{n} + \cos\theta\,(\mathbf{n}\times\mathbf{x})\times\mathbf{n} + \sin\theta\,(\mathbf{n}\times\mathbf{x}). \end{aligned} \tag{6}$$

The point of this last equation is that the action of R on $\mathbb{R}^3$ is completely specified by $\mathbf{n}$ and θ. Accordingly, if two matrices in $SO(3)$ have same $\mathbf{n}$ and same θ, they must be equal.

Now we show that for any $R\in SO(3)$, $\exists q\in S^3 :\ Q(q) = R$. Given a matrix $R\in SO(3)$ we compute θ and $\mathbf{n}$ and build the matrix $Q\big((\cos\theta/2,\,\sin\theta/2\,n_1,\,\sin\theta/2\,n_2,\,\sin\theta/2\,n_3)\big)$. By construction $\theta(R) = \theta(Q)$ and $\mathbf{n}(R) = \mathbf{n}(Q)$ and it must therefore be that $Q = R$.

Finally we prove the following

Propostion 1. *Let p, $q\in S^3$, then*

$$Q(p) = Q(q) \Leftrightarrow p = \pm\, q. \tag{7}$$

Proof.

$$Q(p) = Q(q) \Leftrightarrow \begin{pmatrix} p_1^2 - p_2^2 - p_3^2 + p_0^2 & 2(p_1p_2 - p_3p_0) & 2(p_1p_3 + p_2p_0) \\ 2(p_1p_2 + p_3p_0) & -p_1^2 + p_2^2 - p_3^2 + p_0^2 & 2(-p_1p_0 + p_2p_3) \\ 2(p_1p_3 - p_2p_0) & 2(p_1p_0 + p_2p_3) & -p_1^2 - p_2^2 + p_3^2 + p_0^2 \end{pmatrix} \tag{8}$$

$$= \begin{pmatrix} q_1^2 - q_2^2 - q_3^2 + q_0^2 & 2(q_1q_2 - q_3q_0) & 2(q_1q_3 + q_2q_0) \\ 2(q_1q_2 + q_3q_0) & -q_1^2 + q_2^2 - q_3^2 + q_0^2 & 2(-q_1q_0 + q_2q_3) \\ 2(q_1q_3 - q_2q_0) & 2(q_1q_0 + q_2q_3) & -q_1^2 - q_2^2 + q_3^2 + q_0^2 \end{pmatrix}. \tag{9}$$

The equalities between the diagonal entries imply that $q_i^2 = p_i^2$. Accordingly, there exist four number s_0, s_1, s_2, $s_3\in\{-1,+1\}$ such that $p_i = s_i\,q_i$. Next the equalities between the off-diagonal terms are satisfied iff $s_0 = s_1 = s_2 = s_3$. □

(e) The result comes from computing the 10 possible scalar products between pairs among $\{q, B_1\,q, B_2\,q, B_3\,q\}$.

3 Quaternion manipulation

1. We will use the four equations

$$ijk = -1, \tag{10}$$
$$ii = -1, \tag{11}$$
$$jj = -1, \tag{12}$$
$$kk = -1. \tag{13}$$

Multiplying (10) on the right by k and using (13) there in gives

$$ijk = -1 \Rightarrow ijkk = -k \Rightarrow ij = k. \tag{14}$$

We also find that

$$ijk = -1 \Rightarrow iijk = -i \overset{(11)}{\Rightarrow} jk = i. \tag{15}$$

Finally multiplying (10) on the left by j and on the right by kj gives

$$ijk = -1 \Rightarrow j\, ijk\, kj = -jk\, j \overset{(13,15)}{\Rightarrow} -jijj = -ij \overset{(12)}{\Rightarrow} ji = -ij = -k. \tag{16}$$

Similar arguments allow to complete the ring: $ij = k$, $jk = i$, and $ki = j$ and any pair of non-identical elements in $\{i, j, k\}$ anti-commutes.

2. By direct computations

$$\begin{aligned}
q\,p &= (q_0 + i\,q_1 + j\,q_2 + k\,q_3)\;(p_0 + i\,p_1 + j\,p_2 + k\,p_3),\\
&= q_0p_0 + q_0\,(i\,p_1 + j\,p_2 + k\,p_3) + p_0\,(i\,q_1 + j\,q_2 + k\,q_3) + (i\,q_1 + j\,q_2 + k\,q_3)\,(i\,p_1 + j\,p_2 + k\,p_3),\\
&= q_0p_0 + q_0\,(i\,p_1 + j\,p_2 + k\,p_3) + p_0\,(i\,q_1 + j\,q_2 + k\,q_3)\\
&\quad -q_1p_1 + iq_1(j\,p_2 + k\,p_3) - q_2p_2 + jq_2(i\,p_1 + k\,p_3) - q_3p_3 + kq_3(i\,p_1 + j\,p_2)\\
&= (q_0\,p_0 - \mathbf{q}\cdot\mathbf{p}) + q_0\,(i\,p_1 + j\,p_2 + k\,p_3) + p_0\,(i\,q_1 + j\,q_2 + k\,q_3)\\
&\quad k\,(q_1\,p_2 - q_2p_1) + j\,(q_3p_1 - q_1p_3) + i\,(q_2p_3 - q_3p_2),
\end{aligned} \tag{17}$$

and the result comes from gathering scalar and vector parts.

3. We use equation (10) from the question sheet to show that

$$\begin{aligned}
q\,r\,\overline{q} &= (q_0,\,\mathbf{q})\;(0,\,\mathbf{r})\;(q_0,\,-\mathbf{q}),\\
&= (-\mathbf{q}\cdot\mathbf{r},\,q_0\,\mathbf{r} + \mathbf{q}\times\mathbf{r})\;(q_0,\,-\mathbf{q}),\\
&= \Big(-q_0\,(\mathbf{q}\cdot\mathbf{r}) + (q_0\,\mathbf{r} + \mathbf{q}\times\mathbf{r})\cdot\mathbf{q},\\
&\qquad (\mathbf{q}\cdot\mathbf{r})\,\mathbf{q} + q_0\,(q_0\,\mathbf{r} + \mathbf{q}\times\mathbf{r}) - (q_0\,\mathbf{r} + \mathbf{q}\times\mathbf{r})\times\mathbf{q}\Big)\\
&= \Big(0,\,(\mathbf{q}\otimes\mathbf{q})\,\mathbf{r} + q_0^2\,\mathbf{r} + 2\,q_0\,\mathbf{q}\times\mathbf{r} + \mathbf{q}\times(\mathbf{q}\times\mathbf{r})\Big),\\
&= \Big(0,\,(q_0^2 + \mathbf{q}\cdot\mathbf{q})\,\mathbf{r} + 2\,q_0\,\mathbf{q}\times\mathbf{r} + 2\,\mathbf{q}\times(\mathbf{q}\times\mathbf{r})\Big),
\end{aligned} \tag{18}$$

where we used the fact that $\mathbf{q}\otimes\mathbf{q} = (\mathbf{q}\cdot\mathbf{q})\,\mathrm{Id} + \mathbf{q}^\times\,\mathbf{q}^\times$.

This is identical to the Rodrigues formula whenever $q\overline{q} = 1$. In particular, this shows that if we have a rotation of quaternion q (of norm 1), then the it rotates a vector $\mathbf{r}$ onto the vector part of $q\,r\,\overline{q}$ as discussed during the lecture.