

## 1 Framings of closed curves (revisiting material from last lecture)

### 1. Existence of a closed framing.

- a) First, because the curve is regular, we can re-parameterise it by arc-length so that we are able to apply the results of session 2 (we keep the letter  $s$  as a parameter and simply assume that it is now arc-length). Then we pick an arbitrary smooth function, say  $u_3(s) = \cos s$  and pick a unit vector  $d_{10}$  perpendicular to  $\mathbf{x}'(0)$ . Then we build a framing  $D(s) = \{\mathbf{d}_1(s), \mathbf{d}_2(s), \mathbf{d}_3(s)\}$  of  $\mathbf{x}$  by solving the following initial value problem

$$\begin{aligned} \mathbf{d}'_i(s) &= (u_3(s) \mathbf{d}_3(s) + \mathbf{x}' \times \mathbf{x}'') \times \mathbf{d}_i(s), \\ \mathbf{d}_1(0) &= \mathbf{d}_{10}, \\ \mathbf{d}_2(0) &= \mathbf{x}'(0) \times \mathbf{d}_{10}, \\ \mathbf{d}_3(0) &= \mathbf{x}'(0). \end{aligned} \tag{1}$$

See session 2 for a proof that the solution of (1) is indeed an adapted framing to  $\mathbf{x}$  and that the third component of its Darboux vector is  $u_3$ . In particular, because  $\mathbf{x}$  is smoothly closed, we have  $\mathbf{d}_3(0) = \mathbf{d}_3(L)$ . However, in general there exists an angle  $\phi$  (that depends on the function  $u_3$  and on the shape of the curve) such that

$$D(L) = D(0) R_{3,\phi}. \tag{2}$$

If  $\phi$  is an integer multiply of  $2\pi$ , then we were lucky and our function  $u_3$  provided a closed framing.

- b) If not, we define a new function

$$q_3(s) = u_3(s) - \phi/L. \tag{3}$$

and define an associated framing  $C(s) = \{\mathbf{c}_1(s), \mathbf{c}_2(s), \mathbf{c}_3(s)\}$  as the solution of the IVP

$$\begin{aligned} \mathbf{c}'_i(s) &= (q_3(s) \mathbf{d}_3(s) + \mathbf{x}' \times \mathbf{x}'') \times \mathbf{c}_i(s), \\ \mathbf{c}_1(0) &= \mathbf{d}_{10}, \\ \mathbf{c}_2(0) &= \mathbf{x}'(0) \times \mathbf{d}_{10}, \\ \mathbf{c}_3(0) &= \mathbf{x}'(0). \end{aligned} \tag{4}$$

Note in particular that  $C(0) = D(0)$ .

The register angle between  $D$  and  $C$  is given by

$$\varphi(s) = \int_0^s q_3(s') - u_3(s') ds' = -\phi \frac{s}{L}. \tag{5}$$

In particular, we have

$$C(L) = D(L) R_{3,\varphi(L)} = D(0) = C(0), \tag{6}$$

so that  $C$  is a closed framing.

2. Writhe framings are always closed.

- a) Let  $W(s) = \{\mathbf{w}_1(s), \mathbf{w}_2(s), \mathbf{w}_3(s)\}$  be a Writhe framing with  $\mathbf{w}_1(0) = \mathbf{d}_{10}$  as defined in the previous question. That is, the local twist of  $W(s)$  is  $\omega_3(s) = -\frac{1}{2} \int_0^L \text{wr}(s, \sigma) d\sigma$ . Let  $\varphi(s)$  be the register angle between  $W$  and the closed framing  $C$  from the previous question. We have

$$\varphi(s) = \int_0^s \mathbf{q}_3(s') - \omega_3(s') ds' \quad (7)$$

- b) Accordingly, we have

$$\varphi(L) = 2\pi(Tw(C) + Wr(x)) = 2\pi Lk(\mathbf{x}; C). \quad (8)$$

Hence

$$W(L) = C(L) R_{3, -\varphi_L} = C(L) R_{3, 2k\pi} = C(L) = C(0) = W(0). \quad (9)$$

So that  $W(s)$  is a closed framing.

3. Writhe framings have zero link. Now that we know that  $W(s)$  is closed, we can apply the CFW theorem which readily gives  $Lk = 0$ .

4. a) Any (adapted) framing  $U$  is given by a function  $\mathbf{u}_3(s)$  and an initial condition  $\mathbf{u}_0$ . Let  $\omega_3(s)$  be the third component of the Darboux vector of the writhe framing with initial condition  $\omega_3(0) = \mathbf{u}_0$ . Now we notice that

$$\int_0^L \mathbf{u}_3(s) - \omega(s) ds = \int_0^L \mathbf{u}_s(s) ds + \frac{1}{2} \int_0^L \int_0^L \text{wr}(s, \sigma) d\sigma ds = 2\pi(Tw + Wr) \quad (10)$$

This means that the register angle at  $L$  between the framing  $F$  and the writhe framing (which we know to be closed) is an integer multiple of  $2\pi$ . Therefore the framing  $F$  is also closed.

For the other implication, if the framing  $F$  is closed then we can apply  $Lk = Tw + Wr$  to show that  $Tw + Wr$  is an integer.

- b)  $Wr(\mathbf{x}) = 0 \Rightarrow$  parallel transport framings are closed. Let  $P(s)$  be a parallel transport framing. Then by definition, it has vanishing local twist. Then define  $\psi(s)$  as the register angle between  $P$  and  $W$ , the Writhe framing defined in question 2. Then, we have

$$\psi(L) - \psi(0) = - \int_0^L \omega_3 ds = 2\pi Wr(\mathbf{x}) = 0. \quad (11)$$

Hence  $\psi(L) = \psi(0)$  and since  $W$  is closed so must  $P$ .

- c) The same computation as in 4. shows that  $\psi(L) - \psi(0) = Wr(\mathbf{x})$ . So we have

$$P(L) = W(L) R_{3, \psi(L)} = W(0) R_{3, \psi(L)} = P(0) R_{3, -\psi(0)} R_{3, \psi(L)} = P(0) R_{3, 2\pi Wr(\mathbf{x})}. \quad (12)$$

## 2 Writhe of a figure 8 curve

We start with proving equation (2) of the exercise sheet. We have

$$\mathbf{x}(s) = \begin{pmatrix} \cos s \\ \frac{\sin 2s}{2} \\ \nu \sin s \end{pmatrix}, \quad \text{and therefore} \quad \mathbf{x}'(s) = \begin{pmatrix} -\sin s \\ \cos 2s \\ \nu \cos s \end{pmatrix}. \quad (13)$$

We compute

$$\mathbf{x}(s) - \mathbf{x}(\sigma) = \begin{pmatrix} -2 \sin \frac{s+\sigma}{2} \sin \frac{s-\sigma}{2} \\ \sin(s-\sigma) \cos(s+\sigma) \\ 2\nu \sin \frac{s-\sigma}{2} \cos \frac{s+\sigma}{2} \end{pmatrix} \quad \text{and} \quad \mathbf{x}'(s) \times \mathbf{x}'(\sigma) = \begin{pmatrix} \nu (\cos 2s \cos \sigma - \cos s \cos 2\sigma) \\ \nu \sin(s-\sigma) \\ \sin \sigma \cos 2s - \sin s \cos 2\sigma \end{pmatrix}, \quad (14)$$

and

$$\begin{aligned} \cos 2s \cos \sigma - \cos s \cos 2\sigma &= (2 \cos^2 s - 1) \cos \sigma - \cos s (2 \cos^2 \sigma - 1) \\ &= (\cos s - \cos \sigma) (1 + 2 \cos s \cos \sigma) \\ &= -2 \sin \frac{s+\sigma}{2} \sin \frac{s-\sigma}{2} (1 + 2 \cos s \cos \sigma). \end{aligned} \quad (15)$$

$$\sin \sigma \cos 2s - \sin s \cos 2\sigma = 2 \sin \frac{\sigma-s}{2} \cos \frac{\sigma+s}{2} (1 + 2 \sin \sigma \sin \sigma). \quad (16)$$

Gathering, we find that

$$\begin{aligned} &(\mathbf{x}(s) - \mathbf{x}(\sigma)) \cdot (\mathbf{x}(s) \times \mathbf{x}(\sigma)) \\ &= \nu \left[ 4 \sin^2 \frac{s+\sigma}{2} \sin^2 \frac{s-\sigma}{2} (1 + 2 \cos s \cos \sigma) + \sin^2(s-\sigma) \cos(s+\sigma) \right. \\ &\quad \left. - 4 \sin^2 \frac{s-\sigma}{2} \cos^2 \frac{s+\sigma}{2} (1 + 2 \sin s \sin \sigma) \right], \\ &= 4\nu \sin^2 \frac{s-\sigma}{2} \left[ \sin^2 \frac{s+\sigma}{2} (1 + 2 \cos s \cos \sigma) + \cos^2 \frac{s-\sigma}{2} \cos(s+\sigma) - \cos^2 \frac{s+\sigma}{2} (1 + 2 \sin s \sin \sigma) \right], \\ &= 4\nu \sin^2 \frac{s-\sigma}{2} \left[ \cos^2 \frac{s-\sigma}{2} \cos(s+\sigma) - \cos(s+\sigma) - \cos^2 \frac{s+\sigma}{2} (\cos(s-\sigma) - \cos(s+\sigma)) \right. \\ &\quad \left. + \sin^2 \frac{s+\sigma}{2} (\cos(s-\sigma) + \cos(s+\sigma)) \right], \\ &= 4\nu \sin^2 \frac{s-\sigma}{2} \cos(s+\sigma) \left[ 1 - \sin^2 \frac{s-\sigma}{2} - 1 - \cos(s-\sigma) + 1 \right], \\ &= 4\nu \sin^4 \frac{s-\sigma}{2} \cos(s+\sigma). \end{aligned} \quad (17)$$

We also have

$$\|\mathbf{x}(s) - \mathbf{x}(\sigma)\|^2 = 4 \sin^2 \frac{s-\sigma}{2} \left[ \sin^2 \frac{s+\sigma}{2} + \cos^2 \frac{s-\sigma}{2} \cos^2(s+\sigma) + \nu^2 \cos^2 \frac{s+\sigma}{2} \right]. \quad (18)$$

Gathering (17,18), we find

$$\begin{aligned} Wr(\mathbf{x}) &= \frac{1}{4\pi} \int_{-\pi}^{\pi} \int_{-\pi}^{\pi} \frac{4\nu \sin^4 \frac{s-\sigma}{2} \cos(s+\sigma)}{8 \left| \sin^3 \frac{s-\sigma}{2} \right| \left[ \sin^2 \frac{s+\sigma}{2} + \cos^2 \frac{s-\sigma}{2} \cos^2(s+\sigma) + \nu^2 \cos^2 \frac{s+\sigma}{2} \right]^{3/2}} ds d\sigma, \\ &= \frac{\nu}{4\pi} \int_{-\pi}^{\pi} \int_{-\pi}^{\pi} \frac{|\sin(\frac{\mu}{2} - \sigma)| \cos \mu}{2 \left[ \sin^2 \frac{\mu}{2} + \nu^2 \cos^2 \frac{\mu}{2} + \cos^2(\frac{\mu}{2} - \sigma) \cos^2 \mu \right]^{3/2}} d\mu d\sigma, \\ &= \frac{\nu}{4\pi} \int_{-\pi}^{\pi} \int_0^{\pi} \frac{\sin \tau \cos \mu}{\left[ \sin^2 \frac{\mu}{2} + \nu^2 \cos^2 \frac{\mu}{2} + \cos^2 \tau \cos^2 \mu \right]^{3/2}} d\tau d\mu, \end{aligned} \quad (19)$$

where the second equality comes by changing variables in the inner integral  $\mu = s + \sigma$  and shifting the domain of integration using the fact that the integrand is periodic; the third equality comes after permuting the order of integration and changing variables according to  $\tau = \sigma - \mu/2$ .

It turns out that we can explicitly integrate the inner integral of (19) using

$$\int_0^{\pi} \frac{\sin \tau d\tau}{(A + B \cos^2 \tau)^{3/2}} = \int_{-1}^1 \frac{dx}{(A + Bx^2)^{3/2}} = \frac{1}{A} \left[ \frac{x}{\sqrt{A + Bx^2}} \right]_{-1}^1 = \frac{2}{A\sqrt{A+B}}, \quad (20)$$

where we changed variables according to  $x = \cos \tau$ .

Substituting (20) in (19), we find

$$Wr(\mathbf{x}) = \frac{\nu}{2\pi} \int_{-\pi}^{\pi} \frac{\cos \mu \, d\mu}{\left( \sin^2 \frac{\mu}{2} + \nu^2 \cos^2 \frac{\mu}{2} \right) \sqrt{\sin^2 \frac{\mu}{2} + \cos^2 \mu + \nu^2 \cos^2 \frac{\mu}{2}}}. \quad (21)$$

And we get to (2) from the exercise sheet by noting that the integrand in (21) is even.

Next we want to study the limit of (21) as  $\nu \rightarrow 0$ . To this intend, we will bound the integral above and below and show that both bounds tend to 1 as  $\nu \rightarrow 0^+$ .

1. We find an upper bound to the function  $g(\mu, \nu) = \frac{\cos \mu}{\sqrt{\nu^2 \cos^2 \frac{\mu}{2} + \sin^2 \frac{\mu}{2} + \cos^2 \mu}}$  on the domain  $\mu \in (0, \pi/2)$ . We note that for a given  $\mu$ ,  $g$  is a monotonously decreasing function of  $\nu$ . Hence we have

$$g(\mu, \nu) \leq g(\mu, 1) = \frac{\cos \mu}{\sqrt{1 + \cos^2 \mu}} \leq \frac{\cos \mu}{\sqrt{\cos^2 \mu}} = 1. \quad (22)$$

Also, consider the function  $h(\mu, a) = \frac{1}{\sqrt{a + \cos^2 \mu}}$  with  $a \in (0, 1)$ . We have

$$h(\mu, a) \geq \frac{1}{\sqrt{a + 1}} \geq 1 - a, \quad (23)$$

where the last inequality can be proven by squaring:

$$1 \geq (1 - a)^2(1 + a) \Leftrightarrow 1 \geq (1 - a^2)(1 - a) \Leftrightarrow 1 + a \geq a^2,$$

which is manifestly true for  $a \in (0, 1)$ . Since we have that  $\nu^2 \cos^2 \frac{\mu}{2} + \sin^2 \frac{\mu}{2} \in (0, 1)$ , this proves the inequality (4) of the question sheet.

2. Define  $f(\mu, \nu) = \frac{\cos \mu \, d\mu}{\left( \sin^2 \frac{\mu}{2} + \nu^2 \cos^2 \frac{\mu}{2} \right) \sqrt{\sin^2 \frac{\mu}{2} + \cos^2 \mu + \nu^2 \cos^2 \frac{\mu}{2}}}$ . To make use of the inequalities found in **1.**, we need to split the domain of integration of (2) from the question sheet:

$$Wr(\mathbf{x}; \nu) = \frac{\nu}{\pi} \int_0^{\frac{\pi}{2}} f(\mu, \nu) \, d\mu + \frac{\nu}{\pi} \int_{\frac{\pi}{2}}^{\pi} f(\mu, \nu) \, d\mu.$$

Then we have

$$\lim_{\nu \rightarrow 0^+} Wr(\mathbf{x}; \nu) = \lim_{\nu \rightarrow 0^+} \frac{\nu}{\pi} \int_0^{\frac{\pi}{2}} f(\mu, \nu) \, d\mu + 0. \quad (24)$$

We next we provide upper and lower bounds for the integral in (24) by noting that all factors in  $f$  are positive when  $\mu \in (0, \frac{\pi}{2})$ :

$$\int_0^{\frac{\pi}{2}} \frac{\cos \mu \, d\mu}{\sin^2 \frac{\mu}{2} + \nu^2 \cos^2 \frac{\mu}{2}} - \int_0^{\frac{\pi}{2}} \cos \mu \, d\mu \leq \int_0^{\frac{\pi}{2}} f(\mu, \nu) \, d\mu \leq \int_0^{\frac{\pi}{2}} \frac{d\mu}{\sin^2 \frac{\mu}{2} + \nu^2 \cos^2 \frac{\mu}{2}}, \quad (25)$$

where the first inequality comes from (4) in the question sheet and the second inequality comes from (3) in the question sheet. To conclude, we compute

$$\begin{aligned} \int_0^{\frac{\pi}{2}} \frac{d\mu}{\sin^2 \frac{\mu}{2} + \nu^2 \cos^2 \frac{\mu}{2}} &= \frac{2}{\nu} \int_0^{\frac{\pi}{2}} \frac{1}{1 + \frac{1}{\nu^2} \tan^2 \frac{\mu}{2}} \frac{d\mu}{2\nu \cos^2 \frac{\mu}{2}}, \\ &= \frac{2}{\nu} \int_0^{1/\nu} \frac{1}{1 + x^2} \, dx, \\ &= \frac{2}{\nu} \arctan [1/\nu], \end{aligned} \quad (26)$$

where  $x = \frac{1}{\nu} \tan \mu/2$ .

We also compute

$$\begin{aligned}
 \int_0^{\frac{\pi}{2}} \frac{\cos \mu}{\sin^2 \frac{\mu}{2} + \nu^2 \cos^2 \frac{\mu}{2}} d\mu &= 2 \int_0^1 \frac{(1-y^2) dy}{(1+y^2)(\nu^2+y^2)}, \\
 &= -\frac{4}{1-\nu^2} \int_0^1 \frac{dy}{1+y^2} + 2 \frac{1+\nu^2}{1-\nu^2} \int_0^1 \frac{dy}{\nu^2+y^2}, \\
 &= -\frac{\pi}{1-\nu^2} + 2 \frac{1+\nu^2}{1-\nu^2} \frac{1}{\nu} \arctan[1/\nu],
 \end{aligned} \tag{27}$$

where the first equality comes by changing variables according to  $y = \tan \mu/2$ .

Substituting (25-27) in (24), we find

$$\begin{aligned}
 \lim_{\nu \rightarrow 0^+} \left[ \frac{-\nu}{1-\nu^2} + \frac{1+\nu^2}{1-\nu^2} \frac{2}{\pi} \arctan[1/\nu] - \frac{\nu}{\pi} \right] &\leq \lim_{\nu \rightarrow 0^+} Wr(\mathbf{x}; \nu) \leq \lim_{\nu \rightarrow 0^+} \frac{2}{\pi} \arctan[1/\nu], \\
 \Leftrightarrow \frac{1}{1} &\leq \lim_{\nu \rightarrow 0^+} Wr(\mathbf{x}; \nu) \leq 1,
 \end{aligned}$$

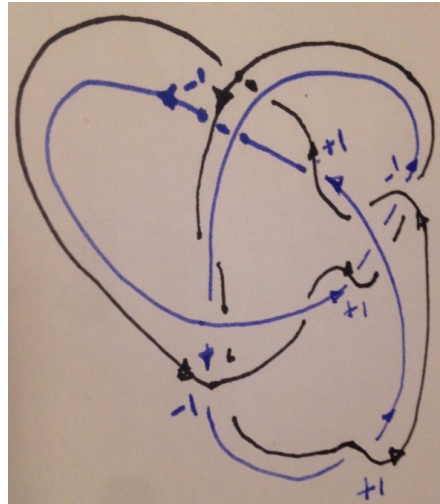
which proves the property

3. The RHS of (2) of the question sheet is an odd function of  $\nu$  so  $\lim_{\nu \rightarrow 0^-} Wr = -1$ .

The dependance of  $Wr$  on  $\nu$  is shown in Fig. 1.

### 3 Writhe Framing of the trefoil

The crossing count develops as follows



### 4 Writhe of an offset curve

1. Equip the curve  $\mathbf{z}(s)$  with the adapted frame  $Z(s) = \{\mathbf{c}_1(s), \mathbf{c}_2(s), \mathbf{c}_3(s)\}$  defined by

$$\mathbf{c}_1 = -\mathbf{d}_1, \quad \mathbf{c}_3 = \frac{\mathbf{z}'}{\|\mathbf{z}'\|}, \quad \text{and} \quad \mathbf{c}_2 = \mathbf{c}_3 \times \mathbf{c}_1. \tag{28}$$

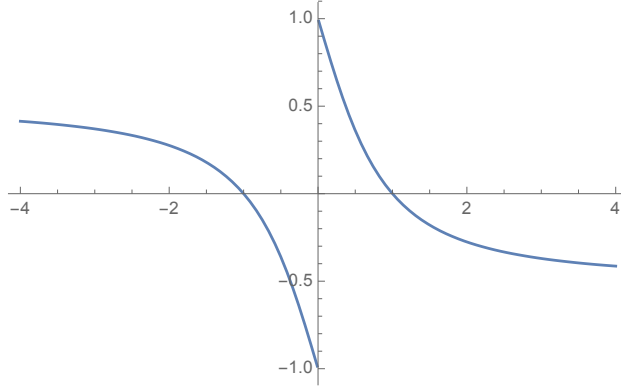


Figure 1: Writhe of the curve  $\mathbf{x}$  of exercise 2 as a function of  $\nu$ .

We also define  $\mathbf{w}(s) = w_i(s) \mathbf{c}_i(s)$  as the Darboux vector of  $Z(s)$  and note that

$$\mathbf{w}_3 = \mathbf{c}'_1 \cdot \mathbf{c}_2. \quad (29)$$

Taking a derivative of (6) from the question sheet, we find

$$\mathbf{z}' = \|\mathbf{z}'\| \mathbf{c}_3 = \|\mathbf{x}'\| \mathbf{d}_3 + \eta \mathbf{d}'_1 \Rightarrow -\mathbf{c}'_1 \stackrel{(28)}{=} \mathbf{d}'_1 = \frac{1}{\eta} \left( \|\mathbf{z}'\| \mathbf{c}_3 - \|\mathbf{x}'\| \mathbf{d}_3 \right). \quad (30)$$

On the other hand, we also have

$$-\mathbf{c}_2 = -\mathbf{c}_3 \times \mathbf{c}_1 \stackrel{(28)}{=} \frac{\mathbf{z}'}{\|\mathbf{z}'\|} \times \mathbf{d}_1 \stackrel{(30)}{=} \left( \frac{\|\mathbf{x}'\|}{\|\mathbf{z}'\|} \mathbf{d}_3 + \eta \frac{\mathbf{d}'_1}{\|\mathbf{z}'\|} \right) \times \mathbf{d}_1. \quad (31)$$

Substituting (30,31) in (29), we find

$$\begin{aligned} \mathbf{w}_3 &= (-\mathbf{c}'_1) \cdot (-\mathbf{c}_2) \stackrel{(30)}{=} \frac{1}{\eta} \left( \|\mathbf{z}'\| \mathbf{c}_3 - \|\mathbf{x}'\| \mathbf{d}_3 \right) \cdot (-\mathbf{c}_2), \\ &= -\frac{\|\mathbf{x}'\|}{\eta} \mathbf{d}_3 \cdot (-\mathbf{c}_2) \stackrel{(31)}{=} -\left( \frac{\|\mathbf{x}'\|}{\eta} \mathbf{d}_3 \right) \cdot \left( \frac{\|\mathbf{x}'\|}{\|\mathbf{z}'\|} \mathbf{d}_3 \times \mathbf{d}_1 + \eta \frac{1}{\|\mathbf{z}'\|} (\mathbf{u} \times \mathbf{d}_1) \times \mathbf{d}_1 \right), \\ &= \frac{\|\mathbf{x}'\|}{\|\mathbf{z}'\|} (\mathbf{u} \times \mathbf{d}_1) \cdot \mathbf{d}_2 = \frac{\|\mathbf{x}'\|}{\|\mathbf{z}'\|} \mathbf{u}_3. \end{aligned} \quad (32)$$

2. We note that equation (6) from the question sheet can be re-written as

$$\mathbf{x} = \mathbf{z} - \eta \mathbf{d}_1 = \mathbf{z} + \eta \mathbf{c}_1, \quad (33)$$

where we can view  $\mathbf{x}$  as an offset of  $\mathbf{z}$  based on the adapted frame  $Z$  to  $\mathbf{z}$ . Next we apply  $Lk = Tw + Wr$  to (6) from the question sheet and to (33) from this sheet. We find

$$\begin{aligned} Lk(\mathbf{x}, \mathbf{z}) &= Wr(\mathbf{x}) + Tw(X) = Wr(\mathbf{x}) + \frac{1}{2\pi} \int_a^b \mathbf{u}_3(s) ds \\ &\parallel \\ Lk(\mathbf{z}, \mathbf{x}) &= Wr(\mathbf{z}) + Tw(Z) = Wr(\mathbf{z}) + \frac{1}{2\pi} \int_a^b \mathbf{w}_3(s) ds. \end{aligned} \quad (34)$$

Substituting (32) in (34) gives the result.