1 Quaternions of particular curves in $SO(3)$

1.1 Quaternion associated with the Frenet frame of a helix

The Frenet frame to $\boldsymbol{\alpha}$ can be recovered from the reference frame $(\mathbf{e}_1\, \mathbf{e}_2\, \mathbf{e}_3)$ by the following sequence of (right) rotations:

$$\begin{pmatrix}\mathbf{n} & \mathbf{b} & \mathbf{t}\end{pmatrix} = \begin{pmatrix}\mathbf{e}_1 & \mathbf{e}_2 & \mathbf{e}_3\end{pmatrix} R_{3,s/c+\pi}\, R_{1,\phi}, \tag{1}$$

where the matrices $R_{i,\alpha}$ were defined in the lecture and where $\phi = \mathrm{Arctan(a/b)}$. In particular,

$$\cos\phi/2 = \sqrt{\frac{c+b}{2c}}, \qquad \text{and} \qquad \sin\phi/2 = \sqrt{\frac{c-b}{2c}}. \tag{2}$$

Hence the quaternion associated to the rotation from the lab frame $(\mathbf{e}_1\, \mathbf{e}_2\, \mathbf{e}_3)$ to the Frenet-Serret frame $(\mathbf{n}\, \mathbf{b}\, \mathbf{t})$ is

$$\begin{aligned} q(s) &= \left[\cos\left(\frac{s}{2c}+\frac{\pi}{2}\right) + k\, \sin\left(\frac{s}{2c}+\frac{\pi}{2}\right)\right]\ \left[\cos\left(\phi/2\right) + i\, \sin\left(\phi/2\right)\right], \\ &= \left[-\sin\left(\frac{s}{2c}\right) + k\, \cos\left(\frac{s}{2c}\right)\right]\ \left[\sqrt{\frac{c+b}{2c}} + i\sqrt{\frac{c-b}{2c}}\right], \\ &= -\sin\frac{s}{2c}\sqrt{\frac{c+b}{2c}} - i\, \sin\frac{s}{2c}\sqrt{\frac{c-b}{2c}} + j\, \cos\frac{s}{2c}\sqrt{\frac{c-b}{2c}} + k\, \cos\frac{s}{2c}\sqrt{\frac{c+b}{2c}}. \end{aligned} \tag{3}$$

1.2 Quaternion of the multiply covered circle

1. We simply take the quaternion of the previous question in the case $b = 0$ and a the radius of the circle:

$$q(s) = \cos\frac{s}{2a}\,\frac{j+k}{\sqrt{2}} - \sin\frac{s}{2a}\,\frac{1+i}{\sqrt{2}}. \tag{4}$$

The fact that the circle is covered n times is encoded in the domain of $s \in [0, 2\pi\, n]$. Also $(j+k)/\sqrt{2}$ and $(1+i)/\sqrt{2}$ are themselves unit quaternions so that the image of $q(s)$ is a great circle in S^3. It is covered $n/2$ times: closed if n is even and open if n is odd.

2. The register angle ψ from the frame of the previous question to a frame with prescribed twist $\mathsf{u}_3(s)$ is

$$\psi(s) = \int_0^s \mathsf{u}_3(\sigma)\, d\sigma + \psi(0). \tag{5}$$

Accordingly the quaternion that we are looking for is

$$\begin{aligned} q(s) &= \left(\cos\frac{s}{2a}\,\frac{j+k}{\sqrt{2}} - \sin\frac{s}{2a}\,\frac{1+i}{\sqrt{2}}\right)\left(\cos\big(\psi(s)/2\big) + k\, \sin\big(\psi(s)/2\big)\right), && (6) \\ &= -\sin\frac{\psi(s)+s/a}{2} + i\, \sin\frac{\psi(s)-s/a}{2} + j\, \cos\frac{\psi(s)-s/a}{2} + k\cos\frac{\psi(s)+s/a}{2}. && (7) \end{aligned}$$

2 Cayley transforms

2.1 A few general properties

1. *Ad absurdum*, if $(I+M)$ is not invertible, then there exists a non-trivial (i.e. $\neq \mathbf{0}$) vector $\mathbf{v}$ such that $(I+M)\,\mathbf{v} = \mathbf{0}$. This implies that

$$\begin{aligned} M\,\mathbf{v} = -\mathbf{v} \quad &\Leftrightarrow \quad (I+N)(I-N)^{-1}\,\mathbf{v} = -\mathbf{v}, && (8)\\ &\Leftrightarrow \quad (I-N)^{-1}(I+N)\,\mathbf{v} = -\mathbf{v}, && (9)\\ &\Leftrightarrow \quad (I+N)\,\mathbf{v} = -(I-N)\,\mathbf{v}, && (10)\\ &\Leftrightarrow \quad \mathbf{v} + N\,\mathbf{v} = -\mathbf{v} + N\,\mathbf{v}, \quad \Leftrightarrow \quad \mathbf{v} = -\mathbf{v}, && (11) \end{aligned}$$

a contradiction.

2. We simply have

$$\begin{aligned} M = (I+N)(I-N)^{-1} \quad &\Leftrightarrow \quad M(I-N) = (I+N) && (12)\\ &\Leftrightarrow \quad M - I = (I+M)N \overset{2.1.1}{\Leftrightarrow} N = (I+M)^{-1}(M-I). && (13) \end{aligned}$$

3. First note that if Q is a Cayley transform, then $(I+Q)$ is invertible and so Q can not have -1 as an eigenvalue. This in turn implies $Q \notin O(n) \setminus SO(n)$ so that if Q is orthogonal, then it must be that $Q \in SO(n)$. Then we have

$$\begin{aligned} Q^T Q = I \quad &\Leftrightarrow \quad (I-S)^{-T}(I+S)^T(I+S)(I-S)^{-1} = I, && (14)\\ &\Leftrightarrow \quad (I+S)^T(I+S) = (I-S)^T(I-S), && (15)\\ &\Leftrightarrow \quad I + S^T + S + S^T S = I - S^T - S + S^T S, && (16)\\ &\Leftrightarrow \quad S^T = -S. && (17) \end{aligned}$$

2.2 The case $n = 3$

1. We simply follow the proof highlighted during the lecture. To find Q using the fact that it is the Cayley transform of $S = \mathbf{u}^{\times}$, we first need to compute $(I-S)^{-1}$.

   Let $P \in SO(3)$ be a right-handed orthonormal basis of $\mathbb{R}^3$ such that the third column of P is parallel to $\mathbf{u}$, then we have

$$P^{-1}SP = \begin{pmatrix} 0 & -u & 0 \\ u & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}, \tag{18}$$

   where $u := \|\mathbf{u}\|$. The matrix $P^{-1}SP$ has characteristic polynomial $-\lambda(\lambda^2 + u^2)$. Applying the Cayley-Hamilton theorem then leads to

$$S^3 = -S\,u^2. \tag{19}$$

   Next we look for three numbers α, β and γ such that

$$\begin{aligned} &(I-S)\,(\alpha\, I + \beta\, S + \gamma\, S^2) = I, && (20)\\ \Leftrightarrow \quad &(\alpha - 1)\, I + (\beta - \alpha)\, S + (\gamma - \beta)\, S^2 - \gamma S^3 = 0, && (21)\\ \overset{(19)}{\Leftrightarrow} \quad &(\alpha - 1)\, I + (\beta - \alpha + \gamma\, u^2)\, S + (\gamma - \beta)\, S^2 = 0. && (22) \end{aligned}$$

The condition (22) is satisfied by

$$\alpha = 1, \qquad \beta = \gamma = \frac{1}{1+u^2}. \tag{23}$$

Before we conclude, note that for any vector $\mathbf{v} \in \mathbb{R}^3$, we have

$$S^2\,\mathbf{v} = \mathbf{u}^\times\,(\mathbf{u}^\times\,\mathbf{v}) = \mathbf{u}^\times\,(\mathbf{u}\times\mathbf{v}) = \mathbf{u}\times(\mathbf{u}\times\mathbf{v}) = \mathbf{u}\,(\mathbf{u}\cdot\mathbf{v}) - u^2\,\mathbf{v} = (\mathbf{u}\otimes\mathbf{u} - u^2 I)\,\mathbf{v}. \tag{24}$$

Finally, compute

$$\begin{aligned}
Q &= (I+S)\,(I-S)^{-1}, && (25)\\
&= (I+S)\,\frac{1}{1+u^2}\Big((1+u^2)\,I + S + S^2\Big), && (26)\\
&= \frac{1}{1+u^2}\Big((1+u^2)\,I + (2+u^2)\,S + 2\,S^2 + S^3\Big) && (27)\\
&\overset{(19)}{=} \frac{1}{1+u^2}\Big((1+u^2)\,I + 2\,S + 2\,S^2\Big), && (28)\\
&\overset{(24)}{=} \frac{1}{1+u^2}\Big((1-u^2)\,I + 2\,\mathbf{u}^\times + 2\,\mathbf{u}\otimes\mathbf{u}\Big). && (29)
\end{aligned}$$

As for the second part of the question, the Rodrigues formula directly gives the matrix of a right-handed rotation of angle ϕ and axis $\mathbf{n}$:

$$R(\phi,\mathbf{n}) = \cos\phi\, I + (1-\cos\phi)\,\mathbf{n}\otimes\mathbf{n} + \sin\phi\,\mathbf{n}^\times. \tag{30}$$

For $\phi = 2\,\mathrm{Arctan}(u)$ with $u > 0$, we have $\tan\frac{\phi}{2} = u$. So that both

$$\cos\phi = \frac{1-\tan^2\frac{\phi}{2}}{1+\tan^2\frac{\phi}{2}} = \frac{1-u^2}{1+u^2}, \quad \text{and} \quad \sin\phi = \frac{2\tan\frac{\phi}{2}}{1+\tan^2\frac{\phi}{2}} = \frac{2u}{1+u^2}. \tag{31}$$

Substituting both (31) and $\mathbf{n} = \frac{1}{u}\mathbf{u}$ in (30) yields

$$R(2\,\mathrm{Arctan}(u), \mathbf{u}/u) = \frac{1}{1+u^2}\left[(1-u^2)\,I + 2\,u^2\,\frac{\mathbf{u}}{u}\otimes\frac{\mathbf{u}}{u} + 2\,u\,\left(\frac{\mathbf{u}}{u}\right)^\times\right] = Q. \tag{32}$$

2. We will use the result of 2.1.2 to find S. To this intend, we first compute $(I+Q)^{-1}$. The characteristic polynomial of a 3×3 matrix A is

$$-\lambda^3 + (\mathrm{tr}\,A)\lambda^2 - \frac{1}{2}\left((\mathrm{tr}\,A)^2 - \mathrm{tr}\,A^2\right)\lambda + \det A. \tag{33}$$

Since $Q \in SO(3)$, the Cayley-Hamilton theorem gives

$$Q^3 - t\,Q^2 + t\,Q - I = 0, \tag{34}$$

where $t := \mathrm{Trace}(Q)$. Multiplying (34) by Q^T yields

$$Q^2 = t\,Q - t\,I + Q^T. \tag{35}$$

Next we try to find three numbers α, β and γ such that

$$(I+Q)(\alpha\,Q^T + \beta\,I + \gamma\,Q) = I. \tag{36}$$

To this intend compute

$$\begin{aligned}
\text{Eq. (36)} \quad &\Leftrightarrow \quad \alpha\, Q^T + (\alpha+\beta-1)\, I + (\gamma+\beta)\, Q + \gamma\, Q^2 = 0, \\
&\overset{(35)}{\Leftrightarrow} \quad (\alpha+\gamma)\, Q^T + (\alpha+\beta-1-t\,\gamma)\, I + (\gamma\,(1+t)+\beta)\, Q = 0, \qquad (37)\\
&\Leftarrow \quad \begin{cases} \alpha+\gamma = 0, \\ \alpha+\beta-t\,\gamma = 1, \\ \beta+(1+t)\,\gamma = 0, \end{cases} \quad \Leftrightarrow \quad \begin{cases} \alpha = \dfrac{1}{2\,(1+t)}, \\ \beta = \dfrac{1+t}{2\,(1+t)}, \\ \gamma = \dfrac{-1}{2\,(1+t)}. \end{cases} \qquad (38)
\end{aligned}$$

Accordingly, the final result in (38) contains the three numbers that we were looking for: we have shown that

$$(I+Q)^{-1} = \frac{1}{2\,(1+t)}\Big(Q^T + (1+t)\, I - Q\Big). \tag{39}$$

Now, making use of the result 2.1.2 of this sheet, we compute

$$\begin{aligned}
S \quad &= \quad (I+Q)^{-1}\,(Q-I) = \frac{1}{2\,(1+t)}\Big(Q^T + (1+t)\, I - Q\Big)\,(Q-I), &(40)\\
&= \quad \frac{1}{2\,(1+t)}\Big(-Q^T - t\, I + (2+t)\, Q - Q^2\Big), &(41)\\
&\overset{(35)}{=} \quad \frac{Q-Q^T}{1+t} &(42)
\end{aligned}$$

3. Substituting $\mathbf{u} = \tan\frac{\phi}{2}\mathbf{u}$ into the formula for Q gives

$$Q = \frac{1-\tan^2\frac{\phi}{2}}{1+\tan^2\frac{\phi}{2}} I + \frac{2\tan^2\frac{\phi}{2}}{1+\tan^2\frac{\phi}{2}}\mathbf{n}\otimes\mathbf{n} + \frac{2\tan\frac{\phi}{2}}{1+\tan^2\frac{\phi}{2}}\mathbf{n}^\times$$

Using the trigonometric identity $\sec^2 x = 1+\tan^2 x$ we get

$$Q = \left(\cos^2\frac{\phi}{2} - \sin^2\frac{\phi}{2}\right) I + \left(2\sin^2\frac{\phi}{2}\right)\mathbf{n}\otimes\mathbf{n} + \left(2\sin\frac{\phi}{2}\cos\frac{\phi}{2}\right)\mathbf{n}^\times$$

Now applying double angle formulae gives the formula from sheet 8

$$Q = \cos\phi\; I + (1-\cos\phi)\;\mathbf{n}\otimes\mathbf{n} + \sin\phi\;\mathbf{n}^\times$$

2.3 Composition of rotations

We have seen in class that if $Q \in SO(3)$, it has eigenvalues $\{1,\, e^{i\theta},\, e^{-i\theta}\}$ for some $\theta \in (-\pi,\, \pi]$. If -1 is not an eigenvalue of Q (that is $-\pi < \theta < \pi$), then the Cayley vector $\mathbf{u}$ of Q satisfies $u := ||\mathbf{u}|| = |\tan\theta/2|$. Also $\mathbf{u}$ is in the eigenspace of the $+1$ eigenvalue of Q. On the other hand, the quaternion associated to Q is

$$q = (q_0,\, \mathbf{q}) = \pm(\cos\theta/2,\, \sin\theta/2\,\mathbf{n}), \tag{43}$$

where $\mathbf{n}$ is in the eigenspace of the eigenvalue $+1$ of Q. It follows that

$$\mathbf{u} = \mathbf{q}/q_0. \tag{44}$$

Now consider the two matrices Q_1 and Q_2 of the exercise. They have associated quaternions q and p and the composition of which reads (see last week's exercise session)

$$
\begin{aligned}
q\,p &= (q_0\,p_0 - \mathbf{q}\cdot\mathbf{p},\; q_0\,\mathbf{p} + p_0\,\mathbf{q} + \mathbf{q}\times\mathbf{p}) \\
&= q_0\,p_0\left(1 - \frac{\mathbf{q}}{q_0}\cdot\frac{\mathbf{p}}{p_0},\; \frac{\mathbf{p}}{p_0} + \frac{\mathbf{q}}{q_0} + \frac{\mathbf{q}}{q_0}\times\frac{\mathbf{p}}{p_0}\right) \\
&= q_0\,p_0\,(1 - \mathbf{u}_1\cdot\mathbf{u}_2,\; \mathbf{u}_1 + \mathbf{u}_2 + \mathbf{u}_1\times\mathbf{u}_2)\,. \qquad (45)
\end{aligned}
$$

Substituting (45) in (44) gives the Cayley vector of the composition as expected.

2.4 Cayley transforms in $SE(3)$

First, we prove that if $\mathcal{S}$ is of the form

$$
\mathcal{S} = \begin{pmatrix} \mathbf{u}^\times & \mathbf{v} \\ 0 & 0 \end{pmatrix}, \qquad (46)
$$

then its Cayley transform is in $SE(3)$. This is because

$$
\begin{aligned}
(\mathcal{I}+\mathcal{S})\,(\mathcal{I}-\mathcal{S})^{-1} &= \begin{pmatrix} I+\mathbf{u}^\times & \mathbf{v} \\ 0 & 1 \end{pmatrix}\begin{pmatrix} I-\mathbf{u}^\times & -\mathbf{v} \\ 0 & 1 \end{pmatrix}^{-1}, \qquad (47) \\
&= \begin{pmatrix} I+\mathbf{u}^\times & \mathbf{v} \\ 0 & 1 \end{pmatrix}\begin{pmatrix} (I-\mathbf{u}^\times)^{-1} & (I-\mathbf{u}^\times)^{-1}\mathbf{v} \\ 0 & 1 \end{pmatrix} \qquad (48) \\
&= \begin{pmatrix} Q & Q\mathbf{v}+\mathbf{v} \\ 0 & 1 \end{pmatrix}, \qquad (49)
\end{aligned}
$$

where $Q := (I + \mathbf{u}^\times)(I - \mathbf{u}^\times)^{-1} \in SO(3)$ because it is the Cayley transform of a skew matrix. Accordingly, the RHS of (49) is in $SE(3)$.

Next, we show that if $\mathcal{Q}$ is the Cayley transform of some matrix $\mathcal{S} \in \mathbb{R}^{4\times4}$ and $\mathcal{Q} \in SE(3)$, then $\mathcal{S}$ is of the form (46).

If $\mathcal{Q} \in SE(3)$ then there exist a matrix $Q \in SO(3)$ and a vector $\mathbf{q}$ such that

$$
\mathcal{Q} = \begin{pmatrix} Q & \mathbf{q} \\ 0 & 1 \end{pmatrix}. \qquad (50)
$$

Furthermore, because $\mathcal{Q}$ is a Cayley transform, $\mathcal{Q}+\mathcal{I}$ must be invertible. This in turn implies that $Q+I$ is invertible. Then after defining $S = (Q+1)^{-1}(Q-1)$ a similar argument to that developed in 2.1.1 (make sure you can do it) shows that $I - S$ is invertible and a similar argument to 2.1.2 shows that Q is the Cayley transform of S. Finally, 2.1.3 ensures that S must be skew so there exists a vector $\mathbf{u}$ such that $S = \mathbf{u}^\times$.

The result 2.1.2 implies

$$
\begin{aligned}
\mathcal{S} &= (\mathcal{Q}+\mathcal{I})^{-1}(\mathcal{Q}-\mathcal{I}) = \begin{pmatrix} Q+I & \mathbf{q} \\ 0 & 2 \end{pmatrix}^{-1}\begin{pmatrix} Q-I & \mathbf{q} \\ 0 & 0 \end{pmatrix} \\
&= \begin{pmatrix} (Q+I)^{-1} & -\frac{1}{2}(Q+I)^{-1}\mathbf{q} \\ 0 & 1/2 \end{pmatrix}\begin{pmatrix} Q-I & \mathbf{q} \\ 0 & 0 \end{pmatrix} = \begin{pmatrix} \mathbf{u}^\times & (Q+I)^{-1}\mathbf{q} \\ 0 & 0 \end{pmatrix}.
\end{aligned}
$$