1 Euler-Rodrigues parameters

(a) As discussed \( q \in S^3 \) is associated to a rotation of angle \( \phi \) and axis \( \mathbf{n} \). However, the rotation matrix \( R(\phi, \mathbf{n}) \in SO(3) \) corresponding to a rotation of angle \( \phi \) and axis along the unit vector \( \mathbf{n} \) is given by the Rodrigues formula:

\[
R(\phi, \mathbf{n}) = \cos \phi \text{Id} + (1 - \cos \phi) \mathbf{n} \otimes \mathbf{n} + \sin \phi \mathbf{n}^\times,
\]

\[
= (\cos^2 \frac{\phi}{2} - \sin^2 \frac{\phi}{2}) \text{Id} + 2 \sin \frac{\phi}{2} \mathbf{n} \otimes \mathbf{n} + 2 \cos \frac{\phi}{2} \sin \frac{\phi}{2} \mathbf{n}^\times,
\]

\[
= (q_0^2 - \mathbf{q} \cdot \mathbf{q}) \text{Id} + 2 \mathbf{q} \otimes \mathbf{q} + 2q_0 \mathbf{q}^\times = Q(q).
\]  

Expanding (1) gives

\[
Q(q) = \begin{pmatrix}
q_0^2 - q_1^2 & 0 & 0 \\
0 & q_0^2 - q_2^2 - q_3^2 & 0 \\
0 & 0 & q_0^2 - q_1^2 - q_2^2 - q_3^2
\end{pmatrix} + 2 \begin{pmatrix}
q_1^2 & q_1 q_2 & q_1 q_3 \\
q_2 q_1 & q_2^2 & q_2 q_3 \\
q_3 q_1 & q_3 q_2 & q_3^2
\end{pmatrix}.
\]  

(b) Simply computing \( Q(q)\mathbf{q} \) with \( Q(q) \) as defined by (3) on the exercise sheet immediately shows \( Q\mathbf{q} = \mathbf{q} \). However, note that it is even simpler to use the first equality in equation (2) of the exercise sheet. If \( \mathbf{q} = 0 \), the \( q \in S^3 \) implies \( q_0 = \pm 1 \) and \( Q = \text{Id} \).

(c) We have shown in question 2 of session 2 that

\[
\text{tr}(Q) = 1 + 2 \cos \theta,
\]

and also computing the Trace as the sum of the diagonal entries in (3) of the question sheet, we find

\[
4q_0^2 - 1 = 1 + 2 \cos \theta \iff q_0 = \pm \cos \frac{\theta}{2}.
\]  

(d) The fact that any \( q \in S^3 \) corresponds to a single \( Q \in SO(3) \) is manifest from equation (3) of the exercise sheet.

Given a rotation matrix \( R \in SO(3) \), define the vector \( \mathbf{r} \in \mathbb{R}^3 \) and the unit vector \( \mathbf{n} \in S^2 \) according to \( \mathbf{r}^\times = R - R^T \) and \( \mathbf{n} = \mathbf{r}/\|\mathbf{r}\| \). Also, define \( \theta \) as the unique solution of \( 1 + 2 \cos \theta = \text{tr}R \) on \([0, \pi]\).

The angle \( \theta \) and the vector \( \mathbf{n} \) completely specify \( R \). That is, if \( R_1 \) and \( R_2 \) have same \( \theta \) and same \( \mathbf{n} \), then \( R_1 = R_2 \). That is because for any vector \( \mathbf{x} \), we have \( \mathbf{x} = (\mathbf{x} \cdot \mathbf{n}) \mathbf{n} + (\mathbf{n} \times \mathbf{x}) \times \mathbf{n} \). The vector \( (\mathbf{n} \times \mathbf{x}) \times \mathbf{n} \) is perpendicular to \( \mathbf{n} \) and we already showed that if \( \mathbf{v} \perp \mathbf{w} \), for \( \mathbf{w} \) along the axis of \( Q \in SO(3) \), then \( Q\mathbf{v} = \cos \theta \mathbf{v} + \sin \theta \mathbf{n} \times \mathbf{v} \) (make sure that you can close the argument). So we have

\[
R \mathbf{x} = R\left[(\mathbf{x} \cdot \mathbf{n}) \mathbf{n} + (\mathbf{n} \times \mathbf{x}) \times \mathbf{n}\right],
\]

\[
= (\mathbf{x} \cdot \mathbf{n}) R \mathbf{n} + R[(\mathbf{n} \times \mathbf{x}) \times \mathbf{n}],
\]

\[
= (\mathbf{x} \cdot \mathbf{n}) \mathbf{n} + \cos \theta (\mathbf{n} \times \mathbf{x}) \times \mathbf{n} + \sin \theta \mathbf{n} \times ((\mathbf{n} \times \mathbf{x}) \times \mathbf{n}),
\]

\[
= (\mathbf{x} \cdot \mathbf{n}) \mathbf{n} + \cos \theta (\mathbf{n} \times \mathbf{x}) \times \mathbf{n} + \sin \theta (\mathbf{n} \times \mathbf{x}).
\]
The point of this last equation is that the action of $R$ on $\mathbb{R}^3$ is completely specified by $\mathbf{n}$ and $\theta$. Accordingly, if two matrices in $SO(3)$ have same $\mathbf{n}$ and same $\theta$, they must be equal.

Now we show that for any $R \in SO(3)$, $\exists q \in S^3$ : $Q(q) = R$. Given a matrix $R \in SO(3)$ we compute $\theta$ and $\mathbf{n}$ and build the matrix $Q((\cos \theta/2, \sin \theta/2 n_1, \sin \theta/2 n_2, \sin \theta/2 n_3))$. By construction $\theta(R) = \theta(Q)$ and $\mathbf{n}(R) = \mathbf{n}(Q)$ and it must therefore be that $Q = R$.

Finally we prove the following

**Propostion 1.** Let $p, q \in S^3$, then

$$Q(p) = Q(q) \iff p = \pm q.$$  \hfill (6)

**Proof.**

$$Q(p) = Q(q) \iff \begin{pmatrix} p_1^2 - p_2^2 - p_3^2 + p_0^2 & 2(p_1p_2 - p_3p_0) & 2(p_1p_3 + p_2p_0) \\ 2(p_1p_2 + p_3p_0) & -p_1^2 + p_2^2 - p_3^2 + p_0^2 & 2(-p_1p_0 + p_2p_3) \\ 2(p_1p_3 - p_2p_0) & 2(p_1p_0 + p_2p_3) & -p_1^2 - p_2^2 + p_3^2 + p_0^2 \end{pmatrix}$$  \hfill (7)

$$= \begin{pmatrix} q_1^2 - q_2^2 - q_3^2 + q_0^2 & 2(q_1q_2 - q_3q_0) & 2(q_1q_3 + q_2q_0) \\ 2(q_1q_2 + q_3q_0) & -q_1^2 + q_2^2 - q_3^2 + q_0^2 & 2(-q_1q_0 + q_2q_3) \\ 2(q_1q_3 - q_2q_0) & 2(q_1q_0 + q_2q_3) & -q_1^2 - q_2^2 + q_3^2 + q_0^2 \end{pmatrix}. \hfill (8)

The equalities between the diagonal entries imply that $q_i^2 = p_i^2$. Accordingly, there exist four number $s_0, s_1, s_2, s_3 \in \{-1, +1\}$ such that $p_i = s_i q_i$. Next the equalities between the off-diagonal terms are satisfied iff $s_0 = s_1 = s_2 = s_3$. \hfill \square

(e) The result comes from computing the 10 possible scalar products between pairs among $\{q, B_1 q, B_2 q, B_3 q\}$.

(f) Let us denote the columns of $Q(q)$ as $\mathbf{d}_1, \mathbf{d}_2, \mathbf{d}_3$ respectively. Moreover, we define $B(q) := \sum_{i=1}^{3} \mathbf{e}_i \otimes B_i q$ and $F(q) := \sum_{i=1}^{3} \mathbf{d}_i \otimes B_i q$ using the matrices $B_i, i = 1, 2, 3$ introduced in (e). Then we compute

$$F(q)B(q)^T = \begin{pmatrix} \sum_{j=1}^{3} \mathbf{d}_j \otimes B_j q \\ \sum_{i=1}^{3} \mathbf{e}_i \otimes B_i q \end{pmatrix}^T = \begin{pmatrix} \sum_{j=1}^{3} \mathbf{d}_j \otimes B_j q \\ \sum_{i=1}^{3} B_i q \otimes \mathbf{e}_i \end{pmatrix}. \hfill (9)$$

In (e) we have shown that $\{q, B_1 q, B_2 q, B_3 q\}$ is an orthonormal basis for $\mathbb{R}^4$ and expression (9) immediately reduces to

$$F(q)B(q)^T = \sum_{k=1}^{3} \mathbf{d}_k \otimes \mathbf{e}_k = Q(q)$$

Finally, by explicit computation, we have

$$B(q) = \begin{pmatrix} -q_1 & q_0 & q_3 & -q_2 \\ -q_2 & -q_3 & q_0 & q_1 \\ -q_3 & q_2 & -q_1 & q_0 \end{pmatrix} \hfill (10)$$

$$+1$$
and
\[ F(q) = \begin{pmatrix} -q_1 & q_0 & -q_3 & q_2 \\ -q_2 & q_3 & q_0 & -q_1 \\ -q_3 & -q_2 & q_1 & q_0 \end{pmatrix}. \tag{12} \]

\[ B(q) = \begin{pmatrix} -q_1 & q_0 & q_3 & -q_2 \\ -q_2 & -q_3 & q_0 & q_1 \\ -q_3 & q_2 & -q_1 & q_0 \end{pmatrix}, \quad F(q) = \begin{pmatrix} -q_1 & q_0 & -q_3 & q_2 \\ -q_2 & q_3 & q_0 & -q_1 \\ -q_3 & -q_2 & q_1 & q_0 \end{pmatrix}. \tag{13} \]

\section{Composition rule for Euler-Rodrigues parameters}

(a)(b) From Session 12 we know that

\[ B(q) = \begin{pmatrix} -q_1 & q_0 & q_3 & -q_2 \\ -q_2 & -q_3 & q_0 & q_1 \\ -q_3 & q_2 & -q_1 & q_0 \end{pmatrix}, \quad F(q) = \begin{pmatrix} -q_1 & q_0 & -q_3 & q_2 \\ -q_2 & q_3 & q_0 & -q_1 \\ -q_3 & -q_2 & q_1 & q_0 \end{pmatrix}. \]

Then we have

\[ B(q)p := (q, B(q)^T)p = \begin{pmatrix} q_0 & -q_1 & -q_2 & -q_3 \\ q_1 & q_0 & -q_3 & q_2 \\ q_2 & q_3 & q_0 & -q_1 \\ q_3 & -q_2 & q_1 & q_0 \end{pmatrix} \begin{pmatrix} p_0 \\ p_1 \\ p_2 \\ p_3 \end{pmatrix}, \]

\[ = p_0 \begin{pmatrix} q_0 \\ q_1 \\ q_2 \\ q_3 \end{pmatrix} + p_1 \begin{pmatrix} -q_1 \\ q_0 \\ q_3 \\ -q_2 \end{pmatrix} + p_2 \begin{pmatrix} -q_2 \\ -q_3 \\ q_0 \\ q_1 \end{pmatrix} + p_3 \begin{pmatrix} -q_3 \\ q_2 \\ -q_1 \\ q_0 \end{pmatrix}, \]

\[ = p_0q + \sum_{i=1}^{3} p_i B_i q. \]

In the same way, replacing $B(q)$ with $F(q)$ we find also

\[ F(q)p := (q, F(q)^T)p = p_0q + \sum_{i=1}^{3} p_i F_i q. \]

(c)

\[ F(q)^T B(q)v = \begin{pmatrix} q^T \\ F(q) \end{pmatrix} (q, B(q)^T)v = \begin{pmatrix} q^T q & q^T B(q)^T \\ F(q)q & F(q)B(q)^T \end{pmatrix} \begin{pmatrix} 0 \\ v \end{pmatrix} = \begin{pmatrix} q^T B(q)^T v \\ F(q)B(q)^T v \end{pmatrix} = \begin{pmatrix} 0 \\ Q(q)v \end{pmatrix}. \tag{14} \]

Notice that the last equality in (14) is derived from properties 1(e) and 1(f) of exercise sheet 12.

(d) For point (c) we can write

\[ \begin{pmatrix} 0 \\ Q(p)Q(p^*)v \end{pmatrix} = F(p)^T B(p)F(p^*)^T B(p^*)v. \]
and using some algebraic manipulation (here is hidden the very computation complexity) we derive the following series of identities:

\[
\mathcal{F}(p)^T \mathcal{B}(p) \mathcal{F}(p^*)^T \mathcal{B}(p^*) v = \mathcal{F}(p)^T \mathcal{F}(p^*)^T \mathcal{B}(p) \mathcal{B}(p^*) v \\
= [\mathcal{F}(p^*) \mathcal{F}(p)]^T \mathcal{B}(p) \mathcal{B}(p^*) v \\
= \mathcal{F}(\mathcal{F}(p^*) p)^T \mathcal{B}(p) \mathcal{B}(p^*) v \\
= \mathcal{F}(\mathcal{F}(p^*) p)^T \mathcal{B}(\mathcal{B}(p) p^*) v \\
= \mathcal{F}(\mathcal{B}(p) p^*)^T \mathcal{B}(\mathcal{B}(p) p^*) v
\]

Applying again point (c) we can finally write

\[
\begin{pmatrix} q \\ Q(p)Q(p^*) v \end{pmatrix} = \begin{pmatrix} 0 \\ Q(p)Q(p^*) v \end{pmatrix}
\]
and consequently \( q = \mathcal{B}(p) p^* = p_0^* p + \sum_{i=1}^3 p_i^* B_i p \).

3 Composition rule for Cayley vectors

During class we have seen that the relation between the Euler parameters \( q \) and the Cayley vector \( c \) parametrizing the same rotation (of axis \( n \) and angle \( \theta \)) can be obtained by mean of the stereographic projection. Namely, we have

\[
q = (q_0, \mathbf{q})^T = \left( \cos \frac{\theta}{2}, \sin \frac{\theta}{2} \mathbf{n} \right)^T, \quad \mathbf{n} = \frac{\mathbf{c}}{\| \mathbf{c} \|}.
\]

Since \( \| \mathbf{c} \| = \tan \left( \frac{\theta}{2} \right) \), then the explicit relation is \( \mathbf{c} = \frac{\mathbf{q}}{q_0} \).

Consequently, if \( Q(q) = \text{Cay}(\mathbf{c}^\times) = \text{Cay}(\mathbf{u}^\times) \text{Cay}(\mathbf{u}^\times) \) with \( Q(p) = \text{Cay}(\mathbf{u}^\times) \) and \( Q(p^*) = \text{Cay}(\mathbf{u}^\times) \), then \( \mathbf{u} = \frac{\mathbf{P}}{p_0} \) and \( \mathbf{u}^* = \frac{\mathbf{P}^*}{p_0^*} \). Moreover, the previous exercise is giving the composition rule for Euler parameter, which is

\[
q = \mathcal{B}(p) p^* = \left( \begin{array}{ccc} p_0 & -p_1 & -p_2 & -p_3 \\ p_1 & p_0 & -p_3 & p_2 \\ p_2 & p_3 & p_0 & -p_1 \\ p_3 & -p_2 & p_1 & p_0 \end{array} \right) \left( \begin{array}{c} p_0^* \\ -p^T \\ p_0^* + p^\times \end{array} \right) = \left( \begin{array}{c} p_0^* + p_0 p^* - \mathbf{p} \cdot \mathbf{p}^* \\ p_0^* + p_0 p^* + \mathbf{p} \times \mathbf{p}^* \end{array} \right)
\]

and the final result is

\[
\mathbf{c} = \frac{q}{q_0} = \frac{p_0^* p + p_0 p^* + \mathbf{p} \times \mathbf{p}^*}{p_0^* p - \mathbf{p} \cdot \mathbf{p}^*} = \frac{\mathbf{u} + \mathbf{u}^* + \mathbf{u} \times \mathbf{u}^*}{1 - \mathbf{u} \cdot \mathbf{u}^*}.
\]

Notice that in the previous computations we always assumed \( q_0, p_0, p_0^* \) different from zero. This is equivalent of saying that that the rotations \( Q(q), Q(p), Q(p^*) \) are not rotations through an angle \( \pi \). We prove this statement for \( Q(q) \), assuming \( p_0, p_0^* \) different from zero. The proof for \( Q(p) \) and \( Q(p^*) \) is analogous.

If we denote with \( \theta \) the angle of rotation of \( Q(q) \), then

\[
q_0 = p_0 p_0^* - \mathbf{p} \cdot \mathbf{p}^* = p_0 p_0^* (1 - \mathbf{u} \cdot \mathbf{u}^*) = \cos \left( \frac{\theta}{2} \right).
\]
Reminding that $\theta \in [0, \pi]$ we have

$$q_0 = 0 \iff 1 - u \cdot u^* = 0 \iff \theta = \pi.$$

### 4 Darboux vector in terms of Euler-Rodrigues parameters

If $v = \begin{pmatrix} 0 \\ v \end{pmatrix}$ with $v \in \mathbb{R}^3$, then we have

$$\begin{pmatrix} 0 \\ Q'(q(s))v \end{pmatrix} = \left[ F(q(s))^T B(q(s)) \right]' \begin{pmatrix} 0 \\ v \end{pmatrix},$$

with

$$\begin{pmatrix} 0 \\ v \end{pmatrix} = \left[ F(q(s))^T B(q(s)) \right]^T \begin{pmatrix} 0 \\ Q(q(s))v \end{pmatrix},$$

$$= B(q(s))^T F(q(s)) \begin{pmatrix} 0 \\ Q(q(s))v \end{pmatrix}.$$

All together reads as

$$\begin{pmatrix} 0 \\ Q'(q(s))v \end{pmatrix} = \left[ F(q(s))^T B(q(s)) \right]' B(q(s))^T F(q(s)) \begin{pmatrix} 0 \\ Q(q(s))v \end{pmatrix},$$

$$= \left[ F(q'(s))^T F(q(s)) + B(q'(s))B(q(s))^T \right] \begin{pmatrix} 0 \\ Q(q(s))v \end{pmatrix},$$

$$= \left[ F \left( \begin{pmatrix} 0 \\ F(q(s))q'(s) \end{pmatrix} \right)^T + B \left( \begin{pmatrix} 0 \\ F(q(s))q'(s) \end{pmatrix} \right) \right] \begin{pmatrix} 0 \\ Q(q(s))v \end{pmatrix},$$

$$= \left[ 2F(q(s))q'(s) \right]^\times Q(q(s))v.$$

We finally obtain the relation

$$Q'(q(s))v = \left[ 2F(q(s))q'(s) \right]^\times Q(q(s))v = Q(q(s)) \left[ 2B(q(s))q'(s) \right]^\times v \Rightarrow u(s) = 2B(q(s))q'(s).$$

### 5 Euler-Rodrigues parameters of particular curves in $SO(3)$

#### 5.1 Euler-Rodrigues parameters associated with the Frenet frame of a helix

The Frenet frame to $\alpha$ can be recovered from the reference frame $(e_1 \ e_2 \ e_3)$ by the following sequence of (right) rotations:

$$\begin{pmatrix} n & b & t \end{pmatrix} = (e_1 \ e_2 \ e_3) R_{3,s/\pi+c} R_{1,\phi}, \quad (15)$$

where the matrices $R_{i,\alpha}$ were defined in the lecture and where $\phi = \arctan(a/b)$. In particular,

$$\cos \phi/2 = \sqrt{\frac{c + b}{2c}}, \quad \text{and} \quad \sin \phi/2 = \sqrt{\frac{c - b}{2c}}. \quad (16)$$
As suggested in the exercise sheet (expression (9)), it is possible to derive the following composition rule for Euler parameters. Let us assume that \( Q(q) = Q(p)Q(p^*) \), then \( q \) can be written in terms of \( p \) and \( p^* \) as

\[
q = p_0^* p + \sum_{i=1}^{3} p_i^* B_i p.
\]  

(17)

Since \((e_1, e_2, e_3)\) is the identity, \( R_{3,s/c+c} = Q(p(s)) \) and \( R_{1,\phi} = Q(p^*) \), with

\[
p(s) = \left( \cos \left( \frac{s}{2c} + \frac{\pi}{2} \right), \ 0, \ 0, \ \sin \left( \frac{s}{2c} + \frac{\pi}{2} \right) \right)^T = \left( -\sin \left( \frac{s}{2c} \right), \ 0, \ 0, \ \cos \left( \frac{s}{2c} \right) \right)^T,
\]

\[
p^* = \left( \cos (\phi/2), \ \sin (\phi/2), \ 0, \ 0 \right)^T = \left( \sqrt{\frac{c+b}{2c}}, \ \sqrt{\frac{c-b}{2c}}, \ 0, \ 0 \right)^T,
\]

then using (17) we find that the Euler parameters associated to the rotation from the lab frame \((e_1, e_2, e_3)\) to the Frenet-Serret frame \((n, b, t)\) is

\[
q(s) = \left( -\sin \frac{s}{2c} \sqrt{\frac{c+b}{2c}}, \ -\sin \frac{s}{2c} \sqrt{\frac{c-b}{2c}}, \ \cos \frac{s}{2c} \sqrt{\frac{c-b}{2c}}, \ \cos \frac{s}{2c} \sqrt{\frac{c+b}{2c}} \right)^T.
\]

5.2 Euler-Rodrigues parameters of the multiply covered circle

1. We simply take the Euler parameters of the previous question in the case \( b = 0 \) and \( a = c \) the radius of the circle:

\[
q(s) = \left( -\sin \frac{s}{2c} \sqrt{2}, \ -\sin \frac{s}{2c} \sqrt{2}, \ \cos \frac{s}{2c} \sqrt{2}, \ \cos \frac{s}{2c} \sqrt{2} \right)^T
\]

\[
q(s) = \cos \frac{s}{2a} \left( 0, \ 0, \ \frac{1}{\sqrt{2}}, \ \frac{1}{\sqrt{2}} \right)^T - \sin \frac{s}{2a} \left( \frac{1}{\sqrt{2}}, \ \frac{1}{\sqrt{2}}, \ 0, \ 0 \right)^T.
\]  

(18)

The fact that the circle is covered \( n \) times is encoded in the domain of \( s \in [0, 2\pi n] \). Also \( \left( 0, \ 0, \ \frac{1}{\sqrt{2}}, \ \frac{1}{\sqrt{2}} \right)^T \) and \( \left( \frac{1}{\sqrt{2}}, \ \frac{1}{\sqrt{2}}, \ 0, \ 0 \right)^T \) are themselves Euler parameters so that the image of \( q(s) \) is a great circle in \( S^3 \). It is covered \( n/2 \) times: closed if \( n \) is even and open if \( n \) is odd.

2. The register angle \( \psi \) from the frame of the previous question to a frame with prescribed twist \( u_3(s) \) is

\[
\psi(s) = \int_0^s u_3(\sigma) \ d\sigma + \psi(0).
\]  

(19)

Accordingly the Euler parameters that we are looking for are given by the composition rule (17), where

\[
p(s) = \left( -\sin \frac{s}{2c} \sqrt{2}, \ -\sin \frac{s}{2c} \sqrt{2}, \ \cos \frac{s}{2c} \sqrt{2}, \ \cos \frac{s}{2c} \sqrt{2} \right)^T
\]

\[
p^*(s) = \left( \cos \left( \psi(s)/2 \right), \ 0, \ 0, \ \sin \left( \psi(s)/2 \right) \right)^T,
\]

in order to obtain

\[
q(s) = \left( -\sin \frac{\psi(s) + s/a}{2}, \ \sin \frac{\psi(s) - s/a}{2}, \ \cos \frac{\psi(s) - s/a}{2}, \ \cos \frac{\psi(s) + s/a}{2} \right)^T.
\]