Differential Geometry of Framed Curves Prof. John Maddocks Session 13: Solutions A. Flynn

1 Ideal string under frictionless contact

As seen during the lecture,

1. $T(s)$ is constant along the string because if we expand (2) from the exercise sheet, take a scalar product with $\mathbf{x}'$ (remembering that in an arc-length parameterisation of $\mathbf{x}$ we have $\mathbf{x}'' \cdot \mathbf{x}' = 0$), and substitute (3) from the exercise sheet therein, we find

$$0 = (T'\mathbf{x}' + T\mathbf{x}'' + \mathbf{f}) \cdot \mathbf{x}' = T'. \tag{1}$$

2. Substituting $\mathbf{x}'' = \kappa\,\mathbf{n}$ and (1) in (2) from the exercise sheet gives the result.

2 Two strings with a single contact line

If the two strings are at equilibrium, then the sum of external forces acting on any piece of material must vanish (otherwise its centre of gravity would start moving and the system would therefore not be at equilibrium).

Because we assume that there is only a single contact line between the two strings, we can define a mapping that associates the sections which are in contact. With s and σ the arclength parameterisations of the strings, we can therefore speak of the function $s(\sigma)$ that gives the section s of string 1 in contact with a particular section σ of string 2.

Then define the following function

$$\boldsymbol{\xi}(\sigma^\star) = T_1\big(s(\sigma^\star)\big)\ \mathbf{x}_1'\big(s(\sigma^\star)\big) + T_2\big(\sigma^\star\big)\ \dot{\mathbf{x}}_2\big(\sigma^\star\big), \tag{2}$$

that is the total force acting by the material of the second string corresponding to $\sigma > \sigma^\star$ and the material of the first string corresponding to $s > s(\sigma^\star)$ on the material corresponding to $\sigma < \sigma^\star$ and $s < s(\sigma^\star)$.

Let $\sigma_1 < \sigma_2$ both in the range of σ. Consider the set of material points that correspond to the material of $\mathbf{x}_2$ spanned by $\sigma \in [\sigma_1, \sigma_2]$ together with the material points of $\mathbf{x}_1$ spanned by $s \in [s(\sigma_1), s(\sigma_2)]$. The total external force acting on this set of material points sums up to

$$\boldsymbol{\xi}(\sigma_2) - \boldsymbol{\xi}(\sigma_1) = \mathbf{0}, \tag{3}$$

where the equality is necessary since we assume that the strings are at equilibrium (cf. the first paragraph).

At equilibrium, we must therefore have $\boldsymbol{\xi}(\sigma_2) = \boldsymbol{\xi}(\sigma_1)$ for all σ_1 and σ_2 so that $\boldsymbol{\xi}$ must be a constant vector.

3 There exist equilibria of pairs of ideal strings with circular helical centrelines

Consider the equilibria of two strings the centrelines of which are give by

$$\begin{cases} \mathbf{x}_1(s) = \left(a\,(R_1+R_2)\,\cos\dfrac{s}{c_1\,(R_1+R_2)},\ a\,(R_1+R_2)\,\sin\dfrac{s}{c_1(R_1+R_2)},\ \dfrac{b}{c_1}\ s\right), \\ \mathbf{x}_2(\sigma) = \left((1-a)\,(R_1+R_2)\,\cos\dfrac{\sigma}{c_2\,(R_1+R_2)},\ (1-a)\,(R_1+R_2)\,\sin\dfrac{\sigma}{c_2(R_1+R_2)},\ \dfrac{b}{c_2}\ \sigma\right), \end{cases} \tag{4}$$

for some values of $a \in [0,1]$ and $b \in \mathbb{R}$ parameterising different double helices and where

$$c_1 = \sqrt{a^2 + b^2}, \qquad \text{and} \qquad c_2 = \sqrt{(1-a)^2 + b^2}. \tag{5}$$

For these sturctures, the contact mapping discussed in question 2 is

$$s(\sigma) = \frac{c_1}{c_2}\,\sigma = \frac{\sqrt{a^2+b^2}}{\sqrt{(1-a)^2+b^2}}\,\sigma. \tag{6}$$

Using question 1, we must have that

$$\begin{cases} T_1\,\kappa_1 \mathbf{n}_1 + \mathbf{f}_1 = \mathbf{0}, \\ T_2\,\kappa_2 \mathbf{n}_2 + \mathbf{f}_2 = \mathbf{0}, \end{cases} \tag{7}$$

where $\mathbf{f}_1$ is the line density of force *per unit* s applied by the string 2 on the string 1 and where $\mathbf{f}_2$ is the line density of force *per unit* σ applied by the string 1 on the string 2. Hence by action reaction, we have

$$\mathbf{f}_2 = -\mathbf{f}_1 \frac{\mathrm{d}s}{\mathrm{d}\sigma} \overset{(6)}{=} -\mathbf{f}_1 \frac{c_1}{c_2}. \tag{8}$$

Aslo, from exercise 1 in session 1, we recall

$$\mathbf{n}_1\big(s(\sigma)\big) = -\mathbf{n}_2(\sigma) = \left(-\cos\frac{s(\sigma)}{c_1(R_1+R_2)},\, -\sin\frac{s(\sigma)}{c_1(R_1+R_2)},\, 0\right), \tag{9}$$

together with

$$\kappa_1 = \frac{a}{c_1^2\,(R_1+R_2)}, \qquad \text{and} \qquad \kappa_2 = \frac{1-a}{c_2^2\,(R_1+R_2)}. \tag{10}$$

Bringing equations (7-10) thogether, we find that

$$\mathbf{f}_1 = -\frac{T_1\,a}{c_1^2\,(R_1+R_2)}\mathbf{n}_1 \overset{(8)}{=} -\frac{c_2}{c_1}\mathbf{f}_2 = \frac{c_2}{c_1}\frac{(1-a)T_2}{c_2^2\,(R_1+R_2)}\mathbf{n}_2 = -\frac{(1-a)T_2}{c_1\,c_2\,(R_1+R_2)}\mathbf{n}_1. \tag{11}$$

Considering the second and last member of (11), we find

$$\frac{T_2}{T_1} = \frac{c_2}{c_1}\frac{a}{1-a} \overset{(5)}{=} \frac{a\sqrt{(1-a)^2+b^2}}{(1-a)\sqrt{a^2+b^2}}. \tag{12}$$

4 The centrelines of a pair of ideal strings in equilibria form generalised helices

By definition of $\boldsymbol{\xi}$, the three vectors $\boldsymbol{\xi}_1 = T_1\,\mathbf{x}_1'$, $\boldsymbol{\xi}_2 = T_2\,\dot{\mathbf{x}}_2$, and $\boldsymbol{\xi}$ are coplanar (in the plane Π). Furthermore, $\boldsymbol{\xi}_i$ has constant norm T_i. Let ϕ_i be the (positive) angle between $\boldsymbol{\xi}_i$ and $\boldsymbol{\xi}$. Let $\hat{\boldsymbol{\xi}}$ and be $\hat{\boldsymbol{\xi}}^\perp$ be unit vectors respectively along $\boldsymbol{\xi}$ and perpendicular to it in the plane Π. The scalar product of (4) from the question sheet with $\hat{\boldsymbol{\xi}}$ and $\hat{\boldsymbol{\xi}}^\perp$ yields

$$\begin{cases} \cos\phi_1\,T_1 + \cos\phi_2\,T_2 = ||\boldsymbol{\xi}||, \\ \sin\phi_1\,T_1 = \sin\phi_2\,T_2. \end{cases} \tag{13}$$

Since (13) can be inverted to find ϕ_1 and ϕ_2, both angles are constant and so are $\mathbf{x}_1' \cdot \boldsymbol{\xi} = \cos\phi_1\,||\boldsymbol{\xi}||$ and $\dot{\mathbf{x}}_2 \cdot \boldsymbol{\xi} = \cos\phi_2\,||\boldsymbol{\xi}||$.

5 The centrelines of a pair of ideal strings in equilibrium are Bertrand mates

We have seen in question 1 that a pair of ideal strings at equilibrium must respect

$$\begin{cases} T_1\,\kappa_1 \mathbf{n}_1 + \mathbf{f}_1 = \mathbf{0}, \\ T_2\,\kappa_2 \mathbf{n}_2 + \mathbf{f}_2 = \mathbf{0}, \end{cases} \tag{14}$$

Furthermore, the single contact line generates the contact mapping $s(\sigma)$, we then have (action-reaction) that

$$\mathbf{f}_1 \frac{\mathrm{d}s}{\mathrm{d}\sigma} = -\mathbf{f}_2 \tag{15}$$

Therefore multiplying the first equation in (14) by $\dot{s}$ and summing implies $\mathbf{n}_2 = -\mathbf{n}_1$.

Also, let $\mathbf{c}(\sigma)$ be the point of contact between the section of the string 2 at σ with the section of the string 1 at $s(\sigma)$. The contact force must be perpendicular to the contact plane (since it is assumed to be frictionless). Because the sections of the strings are assumed circular, this force must therefore be aligned with $\mathbf{x}_2(\sigma) - \mathbf{c}(\sigma)$ and also with $\mathbf{x}_1(s(\sigma)) - \mathbf{c}(\sigma)$. In summary, we have

$$\mathbf{f}_1(s(\sigma)) \parallel \mathbf{f}_2(\sigma) \parallel \mathbf{n}_1(s(\sigma)) \parallel \mathbf{n}_2(\sigma) \parallel \mathbf{x}_2(\sigma) - \mathbf{c}(\sigma) \parallel \mathbf{x}_1(s(\sigma)) - \mathbf{c}(\sigma), \tag{16}$$

where $\parallel$ means "parallel to".

Finally (16) imply that there exists a function $r(\sigma)$ such that

$$\mathbf{x}_2(\sigma) = \mathbf{x}_1(s(\sigma)) + r(\sigma)\,\mathbf{n}_1(s(\sigma)), \qquad \text{and} \qquad \mathbf{x}_1(s(\sigma)) = \mathbf{x}_2(\sigma) + r(\sigma)\,\mathbf{n}_2(\sigma). \tag{17}$$

Equation (17) is exactly the condition for $\mathbf{x}_1$ and $\mathbf{x}_2$ to be Bertrand mates.

6 Bertrand mates

Let $s(t)$ and $\overline{s}(t)$ be the arclength along the curves $\mathbf{x}(t)$ and $\overline{\mathbf{x}}(t)$. From the exercise sheet, we have

$$\overline{\mathbf{x}}(t) = \mathbf{x}(t) + r(t)\,\mathbf{n}(t). \tag{18}$$

1. Taking a derivative of (18) by t, we find

$$\overline{s}'\,\overline{\mathbf{t}} = s'\,\mathbf{t} + r'\mathbf{n} + r\mathbf{n}', \tag{19}$$

where $'$ denotes derivation w.r.t. t and where $\overline{\mathbf{t}}$ and $\mathbf{t}$ are the unit tangents to $\overline{\mathbf{x}}$ and $\mathbf{x}$.

A scalar product of (19) with $\mathbf{n}$ yields $r' = 0$ since $\mathbf{n}$ is a unit vector implies $\mathbf{n}\cdot\mathbf{n}' = 0$ and $\mathbf{x}$ and $\overline{\mathbf{x}}$ share their principal normal so that $0 = \overline{\mathbf{t}}\cdot\overline{\mathbf{n}} = \pm\,\overline{\mathbf{t}}\cdot\mathbf{n}$.

2. Substituting $\mathbf{n}' = s'\,(\tau\,\mathbf{b} - \kappa\,\mathbf{t})$ in (19) and introducing $\gamma = \frac{\overline{s}'}{s'}$ therein, we find

$$\gamma\,\overline{\mathbf{t}} = \mathbf{t} + r\,(\tau\,\mathbf{b} - \kappa\,\mathbf{t}). \tag{20}$$

Taking a derivative of (20) yields

$$\gamma'\,\overline{\mathbf{t}} + \gamma\,\overline{s}'\overline{\kappa}\,\overline{\mathbf{n}} = s'\,\kappa\,\mathbf{n} + r\,\tau'\,\mathbf{b} - r\,\kappa'\,\mathbf{t} - r\,s'\tau^2\,\mathbf{n} - r\,s'\kappa^2\,\mathbf{n}. \tag{21}$$

Taking the scalar product of (21) with $\mathbf{b}$ and with $\mathbf{t}$ yields

$$\begin{cases} \gamma'(\overline{\mathbf{t}}\cdot\mathbf{b}) = r\,\tau', \\ \gamma'\,(\overline{\mathbf{t}}\cdot\mathbf{t}) = -r\,\kappa'. \end{cases} \tag{22}$$

Also taking the scalar product of (20) with $\mathbf{b}$ and $\mathbf{t}$ yields

$$\begin{cases} \gamma(\bar{\mathbf{t}} \cdot \mathbf{b}) = r\,\tau, \\ \gamma\,(\bar{\mathbf{t}} \cdot \mathbf{t}) = 1 - r\,\kappa. \end{cases} \tag{23}$$

Gathering the systems (22,23), we find the differential equation

$$\frac{d(1 - r\,\kappa)}{1 - r\,\kappa} = \frac{d\tau}{\tau}, \tag{24}$$

which can be easily solved with the initial values $\kappa(0) = \kappa_0$ and $\tau(0) = \tau_0$. We find

$$\frac{1 - r\,\kappa_0}{\tau_0}\,\tau(t) + r\,\kappa(t) = 1. \tag{25}$$

3. If $\mathbf{x}$ has two Bertrand mates, then they are at two different distances r_1 and r_2 from $\mathbf{x}$.

Note that (25) can be reorganised as

$$\frac{1}{\tau_0} - \frac{1}{\tau(t)} = r\,\left(\frac{\kappa(t)}{\tau(t)} - \frac{\kappa_0}{\tau_0}\right). \tag{26}$$

But if (26) holds for two values of r, then we must have $\tau(t) = \tau_0$ and $\kappa(t) = \kappa_0$ both constant in which case $\mathbf{x}$ is a circular helix and (26) holds for all values of r.

7 All equilibria are circular helices

We have seen in question 4 that $\mathbf{x}_1$ and $\mathbf{x}_2$ are generalised helices in which case question 7 of session 1 shows that κ_i/τ_i is constant.

We then have seen in question 5 that $\mathbf{x}_1$ and $\mathbf{x}_2$ are Bertrand mates. We finally have seen in question 6 that if a curve has a Bertrand mate, then it must obey (6) of the question sheet[1].

Gathering the facts that

- κ_i/τ_i is constant and
- there is a constant and non-vanishing linear combination between κ_i and τ_i,

implies that κ_i and τ_i are themselves constant.

8 Points of closest approach for double helices (Optional)

The centreline of one helix is given by

$$\mathbf{x}(s) = \left(R\cos s, R\sin s, Rs\tan\left(\frac{\pi}{2} - \frac{\theta}{2}\right)\right)$$

[1] It must also obey (26) which makes what follows even more direct

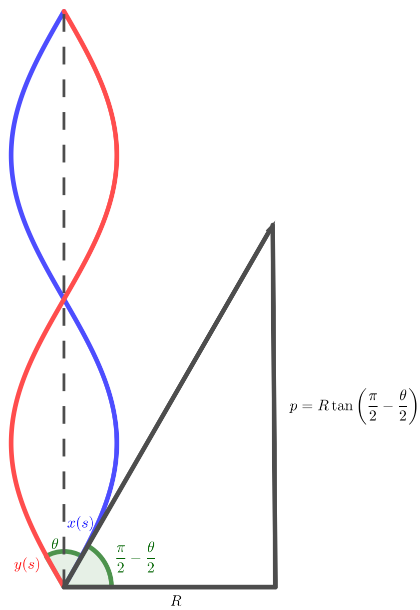

Figure 1: Calculating the pitch of the helices. The view is along the x-axis.

since the pitch of the helix is given by $R\tan\left(\frac{\pi}{2} - \frac{\theta}{2}\right)$, see figure 1. Without loss of generality we calculate the squared distance from $\mathbf{x}(s)$ to $\mathbf{y}(0)$

$$\begin{aligned}
f(s) &= (\mathbf{x}(s) - \mathbf{y}(0)) \cdot (\mathbf{x}(s) - \mathbf{y}(0)) \\
&= (R\cos s + R)^2 + (R\sin s)^2 + \left(Rs\tan\left(\frac{\pi}{2} - \frac{\theta}{2}\right)\right)^2 \\
&= R^2\cos^2 s + 2R^2\cos s + R^2 + R^2\sin^2 s + R^2 s^2\tan^2\left(\frac{\pi}{2} - \frac{\theta}{2}\right) \\
&= R^2 + R^2 + 2R^2\cos s + R^2 s^2\tan^2\left(\frac{\pi}{2} - \frac{\theta}{2}\right)
\end{aligned}$$

Differentiating gives

$$\frac{\mathrm{d}f}{\mathrm{d}s} = -2R^2\sin s + 2R^2 s\tan^2\left(\frac{\pi}{2} - \frac{\theta}{2}\right)$$

The zeros of this function are given by

$$\sin s = -s\tan^2\left(\frac{\pi}{2} - \frac{\theta}{2}\right)$$

This equation always has a solution $s = 0$. This is the only solution if and only if $\tan^2(\frac{\pi}{2} - \frac{\theta}{2}) > 1$ or equivalently $\theta < \frac{\pi}{2}$. For $\theta < \frac{\pi}{2}$, $s = 0$ is a minimum, which can be seen by taking the derivative of $\frac{\mathrm{d}f}{\mathrm{d}s}$. For $\theta > \frac{\pi}{2}$ the turning point at $s = 0$ becomes a local maximum and the two turning points either side of $s = 0$ are the global minima. In this situation the tube is intersecting itself.