1 Ideal string under frictionless contact

As seen during the lecture,

1. \( T(s) \) is constant along the string because if we expand (2) from the exercise sheet, take a scalar product with \( x' \) (remembering that in an arc-length parameterisation of \( x \) we have \( x'' \cdot x' = 0 \)), and substitute (3) from the exercise sheet therein, we find

\[
0 = (T' x' + T x'' + f) \cdot x' = T'. \tag{1}
\]

2. Substituting \( x'' = \kappa n \) and (1) in (2) from the exercise sheet gives the result.

2 Equilibria of two strings with a single contact line

1. The total force acting across the two cross-sections is constant

If the two strings are at equilibrium, then the sum of external forces acting on any piece of material must vanish (otherwise its centre of gravity would start moving and the system would therefore not be at equilibrium).

Because we assume that there is only a single contact line between the two strings, we can define a mapping that associates the sections which are in contact. With \( s \) and \( \sigma \) the arclength parameterisations of the strings, we can therefore speak of the function \( s(\sigma) \) that gives the section \( s \) of string 1 in contact with a particular section \( \sigma \) of string 2.

Then define the following function

\[
\xi(\sigma) = T_1(s(\sigma^*)) x_1'(s(\sigma^*)) + T_2(\sigma^*) x_2'(\sigma^*), \tag{2}
\]

that is the total force acting by the material of the second string corresponding to \( \sigma > \sigma^* \) and the material of the first string corresponding to \( s > s(\sigma^*) \) on the material corresponding to \( \sigma < \sigma^* \) and \( s < s(\sigma^*) \).

Let \( \sigma_1 < \sigma_2 \) both in the range of \( \sigma \). Consider the set of material points that correspond to the material of \( x_2 \) spanned by \( \sigma \in [\sigma_1, \sigma_2] \) together with the material points of \( x_1 \) spanned by \( s \in [s(\sigma_1), s(\sigma_2)] \). The total external force acting on this set of material points sums up to

\[
\xi(\sigma_2) - \xi(\sigma_1) = 0, \tag{3}
\]

where the equality is necessary since we assume that the strings are at equilibrium (cf. the first paragraph).

At equilibrium, we must therefore have \( \xi(\sigma_2) = \xi(\sigma_1) \) for all \( \sigma_1 \) and \( \sigma_2 \) so that \( \xi \) must be a constant vector.

2. There exist equilibria of pairs of ideal strings with circular helical centrelines

Consider the equilibria of two strings the centrelines of which are given by

\[
\begin{align*}
x_1(s) &= \left( a (R_1 + R_2) \cos \frac{s}{c_1(R_1 + R_2)}, a (R_1 + R_2) \sin \frac{s}{c_1(R_1 + R_2)}, \frac{b}{c_1} s \right), \\
x_2(\sigma) &= \left( (1 - a) (R_1 + R_2) \cos \frac{\sigma}{c_2(R_1 + R_2)}, (1 - a) (R_1 + R_2) \sin \frac{\sigma}{c_2(R_1 + R_2)}, \frac{b}{c_2} \sigma \right),
\end{align*} \tag{4}
\]
for some values of $a \in [0, 1]$ and $b \in \mathbb{R}$ parameterising different double helices and where

$$c_1 = \sqrt{a^2 + b^2}, \quad \text{and} \quad c_2 = \sqrt{(1-a)^2 + b^2}. \quad (5)$$

For these structures, the contact mapping discussed in question 2 is

$$s(\sigma) = \frac{c_1}{c_2} \sigma = \frac{\sqrt{a^2 + b^2}}{\sqrt{(1-a)^2 + b^2}}. \quad (6)$$

Using question 1, we must have that

$$\begin{cases}
T_1 \kappa_1 n_1 + f_1 = 0, \\
T_2 \kappa_2 n_2 + f_2 = 0,
\end{cases} \quad (7)$$

where $f_1$ is the line density of force per unit $s$ applied by the string 2 on the string 1 and where $f_2$ is the line density of force per unit $\sigma$ applied by the string 1 on the string 2. Hence by action reaction, we have

$$f_2 = -f_1 \frac{ds}{d\sigma} = -f_1 \frac{c_1}{c_2}. \quad (8)$$

Aslo, from exercise 1 in session 1, we recall

$$n_1(s(\sigma)) = -n_2(\sigma) = \left(-\cos \frac{s(\sigma)}{c_1(R_1 + R_2)}, -\sin \frac{s(\sigma)}{c_1(R_1 + R_2)}, 0 \right), \quad (9)$$

together with

$$k_1 = \frac{a}{c_1^2(R_1 + R_2)}, \quad \text{and} \quad k_2 = \frac{1-a}{c_2^2(R_1 + R_2)}. \quad (10)$$

Bringing equations (7-10) together, we find that

$$f_1 = -\frac{T_1 a}{c_1^2(R_1 + R_2)} n_1 = -\frac{c_2}{c_1} f_2 = \frac{c_2}{c_1} \frac{(1-a)T_2}{c_2^2(R_1 + R_2)} n_2 = -\frac{(1-a)T_2}{c_1 c_2 (R_1 + R_2)} n_1. \quad (11)$$

Considering the second and last member of (11), we find

$$\frac{T_2}{T_1} = \frac{c_2}{c_1} \frac{a}{1-a} \frac{a \sqrt{(1-a)^2 + b^2}}{(1-a) \sqrt{a^2 + b^2}}. \quad (12)$$

3. The centrelines of a pair of ideal strings in equilibria form generalised helices

By definition of $\xi$, the three vectors $\xi_1 = T_1 x_1', \xi_2 = T_2 x_2$, and $\xi$ are coplanar (in the plane $\Pi$). Furthermore, $\xi_i$ has constant norm $T_i$. Let $\phi_i$ be the (positive) angle between $\xi_i$ and $\xi$. Let $\hat{\xi}$ and be $\hat{\xi}^\perp$ be unit vectors respectively along $\xi$ and perpendicular to it in the plane $\Pi$. The scalar product of (4) from the question sheet with $\hat{\xi}$ and $\hat{\xi}^\perp$ yields

$$\begin{cases}
\cos \phi_1 T_1 + \cos \phi_2 T_2 = ||\xi||, \\
\sin \phi_1 T_1 = \sin \phi_2 T_2.
\end{cases} \quad (13)$$

Since (13) can be inverted to find $\phi_1$ and $\phi_2$, both angles are constant and so are $x_1' \cdot \xi = \cos \phi_1 ||\xi||$ and $x_2 \cdot \xi = \cos \phi_2 ||\xi||$. 

4. **The centrelines of a pair of ideal strings in equilibrium are Bertrand mates**

We have seen in question 1 that a pair of ideal strings at equilibrium must respect

\[
\begin{cases}
T_1 \kappa_1 n_1 + f_1 = 0, \\
T_2 \kappa_2 n_2 + f_2 = 0,
\end{cases}
\]  

(14)

Furthermore, the single contact line generates the contact mapping \( s(\sigma) \), we then have (action-reaction) that

\[ f_1 \frac{ds}{d\sigma} = -f_2 \]

(15)

Therefore multiplying the first equation in (14) by \( \dot{s} \) and summing implies \( n_2 = -n_1 \).

Also, let \( c(\sigma) \) be the point of contact between the section of the string 2 at \( \sigma \) with the section of the string 1 at \( s(\sigma) \). The contact force must be perpendicular to the contact plane (since it is assumed to be frictionless). Because the sections of the strings are assumed circular, this force must therefore be aligned with \( x_2(\sigma) - c(\sigma) \) and also with \( x_1(s(\sigma)) - c(\sigma) \). In summary, we have

\[
f_1(s(\sigma)) \parallel n_1(s(\sigma)) \parallel n_2(\sigma) \parallel x_2(\sigma) - c(\sigma) \parallel x_1(s(\sigma)) - c(\sigma),
\]  

(16)

where \( \parallel \) means “parallel to”.

Finally (16) imply that there exists a function \( r(\sigma) \) such that

\[
x_2(\sigma) = x_1(s(\sigma)) + r(\sigma) n_1(s(\sigma)), \quad \text{and} \quad x_1(s(\sigma)) = x_2(\sigma) + r(\sigma) n_2(\sigma).
\]  

(17)

Equation (17) is exactly the condition for \( x_1 \) and \( x_2 \) to be Bertrand mates.

5. **All equilibria are circular helices**

We have seen in question 4 that \( x_1 \) and \( x_2 \) are generalised helices in which case question 7 of session 1 shows that \( \kappa_i/\tau_i \) is constant.

We then have seen in question 5 that \( x_1 \) and \( x_2 \) are Bertrand mates. We finally have seen in question 4 of Session 13 that if a curve has a Bertrand mate, then it must obey (2) of question sheet 13\(^1\).

Gathering the facts that

- \( \kappa_i/\tau_i \) is constant and
- there is a constant and non-vanishing linear combination between \( \kappa_i \) and \( \tau_i \),

implies that \( \kappa_i \) and \( \tau_i \) are themselves constant.

\(^1\)It must also obey (10) of solution sheet 13, which makes what follows even more direct