Differential Geometry of Framed Curves
PROF. JOHN MADDOCKS
SESSION 1 T. LESSINNES

Unless explicitly stated, $\boldsymbol{\alpha} : I \to \mathbb{R}^3$ is a sufficiently smooth curve parameterised by arc length s, with curvature $\kappa(s) > 0$, for all $s \in I$. The vectors $\mathbf{t}$, $\mathbf{n}$ and $\mathbf{b}$ are respectively the unit tangent, normal and binormal to the curve at hand and τ its torsion, and you can assume the Serret Frenet equations. Regular means $\|\boldsymbol{\alpha}'\| \neq 0$. Note that it is usually a good idea to draw a picture.

1. Given the parameterised curve (a circular helix)

$$\boldsymbol{\alpha}(s) = \left(a\cos\frac{s}{c},\ a\sin\frac{s}{c},\ \frac{b}{c}s \right), \qquad s \in \mathbb{R},$$

where $c^2 = a^2 + b^2$,

a) Show that the parameter s is the arc length.

b) Determine the curvature and the torsion of $\boldsymbol{\alpha}$.

c) Determine the Frenet frame of $\boldsymbol{\alpha}$.

d) Show that the straight lines containing the normal $\mathbf{n}(s)$ and passing through $\boldsymbol{\alpha}(s)$ meet the z axis under a constant angle equal to $\pi/2$.

e) Show that the tangent lines to $\boldsymbol{\alpha}$ make a constant angle with the z axis.

2. Given a curve $\boldsymbol{\alpha}(s) : [0, L] \to I$, its Frenet frame $(\mathbf{n}\ \mathbf{b}\ \mathbf{t})$ and its curvature κ and torsion τ. What becomes of $\mathbf{t}$, $\mathbf{b}$, $\mathbf{n}$, κ and τ under the change of parameterisation $s \to L - s$?

3. By construction $\kappa(s) = \|\boldsymbol{\alpha}''\|$. Show that the torsion is

$$\tau(s) = \frac{(\boldsymbol{\alpha}'(s) \wedge \boldsymbol{\alpha}''(s)) \cdot \boldsymbol{\alpha}'''(s)}{\|\boldsymbol{\alpha}''(s)\|^2}.$$

Exercises 4,5 and 6 are for planar curves but each of them generalise to space curves (in $\mathbb{R}^3$) as well.

4. Assume that $\boldsymbol{\alpha}(I) \subset \mathbb{R}^2$ such that $\forall s \in I : \kappa(s) > 0$. Transport the vectors $\mathbf{t}(s)$ in such a way that, for any $s \in I$, the origin of $\mathbf{t}(s)$ agrees with the origin of $\mathbb{R}^2$; the end points of $\mathbf{t}(s)$ then describe a parameterised curve $s \to \mathbf{t}(s)$ called the *indicatrix of tangents* of $\boldsymbol{\alpha}$. Let $\theta(s)$ be the angle from $\mathbf{e}_1$ to $\mathbf{t}(s)$. Prove

a) The indicatrix of tangents is a regular parameterised curve.

b) $\frac{\mathrm{d}\mathbf{t}}{\mathrm{d}s} = \left|\frac{\mathrm{d}\theta}{\mathrm{d}s}\mathbf{n}\right|$, that is, $\kappa = \left|\frac{\mathrm{d}\theta}{\mathrm{d}s}\right|$.

5. Let $\boldsymbol{\alpha} : I \to \mathbb{R}^2$ be a regular plane curve (arbitrary parameter). Assume that $\kappa(t) \neq 0, t \in I$. In this situation, the curve

$$\boldsymbol{\beta}(t) = \boldsymbol{\alpha}(t) + \frac{\mathbf{n}(t)}{\kappa(t)}, \qquad t \in I,$$

is called the *evolute* of $\boldsymbol{\alpha}$.

a) Show that the tangent at t of the evolute of $\boldsymbol{\alpha}$ is the normal to $\boldsymbol{\alpha}$ at t.

b) Consider the normal lines of $\boldsymbol{\alpha}$ at two neighbouring points $t_1 \neq t_2$. Let t_1 approach t_2 and show that the intersection points of the normals converge to a point on the trace of the evolute of $\boldsymbol{\alpha}$.

6. Given a differentiable function $\kappa(s)$, $s \in I$, show that any parameterised plane curve of curvature κ is of the form

$$\boldsymbol{\alpha}(s) = \left(\int_0^s \cos\theta(\sigma)\mathrm{d}\sigma + a, \int_0^s \sin\theta(\sigma)\mathrm{d}\sigma + b \right),$$

where

$$\theta(s) = \int_0^s k(\sigma)\mathrm{d}\sigma + \varphi.$$

Deduce that the curve is determined up to a global translation and rotation.

7. In general, a curve $\boldsymbol{\alpha}$ is called a (noncircular) *helix* if the tangent lines of $\boldsymbol{\alpha}$ make a constant angle with a fixed direction. Assume that $\tau(s) \neq 0$, $s \in I$, and prove the following statements.

a) $\boldsymbol{\alpha}$ is a helix if and only if the lines containing $\mathbf{n}(s)$ and passing through $\boldsymbol{\alpha}(s)$ are parallel to a fixed plane.

b) $\boldsymbol{\alpha}$ is a helix if and only if the lines containing $\mathbf{b}(s)$ and passing through $\boldsymbol{\alpha}(s)$ make a constant angle with a fixed direction.

c) $\boldsymbol{\alpha}$ is a helix if and only if $\kappa/\tau = \text{const.}$

d) The curve

$$\boldsymbol{\alpha}(s) = \left(\frac{a}{c} \int_0^s \sin\theta(\sigma)\mathrm{d}\sigma,\ \frac{a}{c} \int_0^s \cos\theta(\sigma)\mathrm{d}\sigma, \frac{b}{c}s \right),$$

where $c^2 = a^2 + b^2$, is a helix and $\kappa/\tau = a/b$.

8. Assume that $\tau(s) \neq 0$ and $\kappa'(s) \neq 0$ for all $s \in I$. Show that a necessary and sufficient condition for $\boldsymbol{\alpha}(I)$ to lie on a sphere is that

$$R^2 + (R')^2 T^2 = \text{const.},$$

where $R = 1/\kappa$, $T = 1/\tau$, and R' is the derivative of R relative to s.

9. Consider the map

$$\boldsymbol{\alpha}(t) = \begin{cases} (t, e^{-1/t^2}, 0) & t < 0, \\ (0, 0, 0) & t = 0, \\ (t, 0, e^{-1/t^2}) & t > 0. \end{cases} \tag{1}$$

a) Prove that $\boldsymbol{\alpha}$ is a differentiable curve.

b) Prove that $\boldsymbol{\alpha}$ is regular for all t and that the curvature $k(t) \neq 0$, for $t \neq 0$, $t \neq \pm\sqrt{2/3}$, and $k(0) = 0$.

c) For each value of the parameter t the plane spanned by the tangent and normal to the curve at t is called the *osculating* plane. Show that the limit of the osculating planes as $t \to 0^+$ is the plane $y = 0$ but that the limit of the osculating planes as $t \to 0^-$ is the plane $z = 0$. Note that this implies that the normal vector is discontinuous at $t = 0$.

d) Show that $\tau = 0$, even though $\boldsymbol{\alpha}$ is not a plane curve.