Unless explicitly stated, \( \alpha : I \to \mathbb{R}^3 \) is a sufficiently smooth curve parameterised by arc length \( s \), with curvature \( \kappa(s) > 0 \), for all \( s \in I \). The vectors \( t, n \) and \( b \) are respectively the unit tangent, normal and binormal to the curve at hand and \( \tau \) its torsion, and you can assume the Serret Frenet equations. Regular means \( \|\alpha'\| \neq 0 \). Note that it is usually a good idea to draw a picture.

1. Given the parameterised curve (a circular helix)
   \[
   \alpha(s) = \left( a \cos \frac{s}{c}, a \sin \frac{s}{c}, b s \right), \quad s \in \mathbb{R},
   \]
   where \( c^2 = a^2 + b^2 \),
   a) Show that the parameter \( s \) is the arc length.
   b) Determine the curvature and the torsion of \( \alpha \).
   c) Determine the Frenet frame of \( \alpha \).
   d) Show that the straight lines containing the normal \( n(s) \) and passing through \( \alpha(s) \) meet the \( z \) axis under a constant angle equal to \( \pi/2 \).
   e) Show that the tangent lines to \( \alpha \) make a constant angle with the \( z \) axis.

2. Given a curve \( \alpha(s) : [0, L] \to I \), its Frenet frame (\( n b t \)) and its curvature \( \kappa \) and torsion \( \tau \). What becomes of \( t, b, n, \kappa \) and \( \tau \) under the change of parameterisation \( s \to L - s \)?

3. By construction \( \kappa(s) = \|\alpha''\| \). Show that the torsion is
   \[
   \tau(s) = \frac{(\alpha'(s) \wedge \alpha''(s)) \cdot \alpha'''(s)}{\|\alpha''(s)\|^2}.
   \]

Exercises 4, 5 and 6 are for planar curves but each of them generalise to space curves (in \( \mathbb{R}^3 \)) as well.

4. Assume that \( \alpha(I) \subset \mathbb{R}^2 \) such that \( \forall s \in I : \kappa(s) > 0 \). Transport the vectors \( t(s) \) in such a way that, for any \( s \in I \), the origin of \( t(s) \) agrees with the origin of \( \mathbb{R}^2 \); the end points of \( t(s) \) then describe a parameterised curve \( s \to t(s) \) called the indicatrix of tangents of \( \alpha \). Let \( \theta(s) \) be the angle from \( e_1 \) to \( t(s) \). Prove
   a) The indicatrix of tangents is a regular parameterised curve.
   b) \( \frac{dt}{ds} = \left| \frac{d\theta}{ds} \right| n \), that is, \( \kappa = \left| \frac{d\theta}{ds} \right| \).

5. Let \( \alpha : I \to \mathbb{R}^2 \) be a regular plane curve (arbitrary parameter). Assume that \( \kappa(t) \neq 0, t \in I \). In this situation, the curve
   \[
   \beta(t) = \alpha(t) + \frac{n(t)}{\kappa(t)}, \quad t \in I,
   \]
   is called the evolute of \( \alpha \).
   a) Show that the tangent at \( t \) of the evolute of \( \alpha \) is the normal to \( \alpha \) at \( t \).
   b) Consider the normal lines of \( \alpha \) at two neighbouring points \( t_1 \neq t_2 \). Let \( t_1 \) approach \( t_2 \) and show that the intersection points of the normals converge to a point on the trace of the evolute of \( \alpha \).
6. Given a differentiable function $\kappa(s), s \in I$, show that any parameterised plane curve of curvature $\kappa$ is of the form

$$\alpha(s) = \left( \int_0^s \cos \theta(\sigma) d\sigma + a, \int_0^s \sin \theta(\sigma) d\sigma + b \right),$$

where

$$\theta(s) = \int_0^s k(\sigma) d\sigma + \varphi.$$

Deduce that the curve is determined up to a global translation and rotation.

7. In general, a curve $\alpha$ is called a (noncircular) helix if the tangent lines of $\alpha$ make a constant angle with a fixed direction. Assume that $\tau(s) \neq 0, s \in I$, and prove the following statements.

a) $\alpha$ is a helix if and only if the lines containing $n(s)$ and passing through $\alpha(s)$ are parallel to a fixed plane.

b) $\alpha$ is a helix if and only if the lines containing $b(s)$ and passing through $\alpha(s)$ make a constant angle with a fixed direction.

c) $\alpha$ is a helix if and only if $\kappa/\tau = \text{const.}$

d) The curve

$$\alpha(s) = \left( \frac{a}{c} \int_0^s \sin \theta(\sigma) d\sigma, \frac{a}{c} \int_0^s \cos \theta(\sigma) d\sigma, \frac{b}{c} s \right),$$

where $c^2 = a^2 + b^2$, is a helix and $\kappa/\tau = a/b$.

8. Assume that $\tau(s) \neq 0$ and $\kappa'(s) \neq 0$ for all $s \in I$. Show that a necessary and sufficient condition for $\alpha(I)$ to lie on a sphere is that

$$R^2 + (R')^2T^2 = \text{const.},$$

where $R = 1/\kappa, T = 1/\tau$, and $R'$ is the derivative of $R$ relative to $s$.

9. Consider the map

$$\alpha(t) = \begin{cases} 
(t, e^{-1/t^2}, 0) & t < 0, \\
(0, 0, 0) & t = 0, \\
(t, 0, e^{-1/t^2}) & t > 0.
\end{cases} \quad (1)$$

a) Prove that $\alpha$ is a differentiable curve.

b) Prove that $\alpha$ is regular for all $t$ and that the curvature $k(t) \neq 0$, for $t \neq 0, t \neq \pm \sqrt{2/3}$, and $k(0) = 0$.

c) For each value of the parameter $t$ the plane spanned by the tangent and normal to the curve at $t$ is called the osculating plane. Show that the limit of the osculating planes as $t \to 0^+$ is the plane $y = 0$ but that the limit of the osculating planes as $t \to 0^-$ is the plane $z = 0$. Note that this implies that the normal vector is discontinuous at $t = 0$.

d) Show that $\tau = 0$, even though $\alpha$ is not a plane curve.