Differential Geometry of Framed Curves
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Session 2 T. Lessinnes

For question 3 make sure you have completed exercise 3 from last week.

1. Given an orthonormal basis $\{\mathbf{e}_1, \mathbf{e}_2, \mathbf{e}_3\}$ and a particular vector $\mathbf{u}$ in $\mathbb{R}^3$, denote by $\mathrm{Sk}(\mathbf{u})$ the operator that gives $\mathrm{Sk}(\mathbf{u})\mathbf{v} = \mathbf{u} \wedge \mathbf{v}$. Show that for any invertible matrix $M \in \mathbb{R}^3 \times \mathbb{R}^3$,

$$\mathrm{Sk}(M\mathbf{u}) = |M|\, M^{-T}\, \mathrm{Sk}(\mathbf{u})\, M^{-1}.$$

Show that if $M \in \mathrm{SO}(3)$ then this formula simplifies to

$$\mathrm{Sk}(M\mathbf{u}) = M\mathrm{Sk}(\mathbf{u})M^{-1}$$

Now let $\tilde{\mathbf{u}}$ be the Darboux vector of a curve $R \in SO(3)$ so $\mathbf{d}_i' = \tilde{\mathbf{u}} \wedge \mathbf{d}_i$ where $\mathbf{d}_i$ are the columns of R and let $\mathbf{v}$ be a vector such that $\tilde{\mathbf{u}} = R\mathbf{v}$. Show that

$$R' = \mathrm{Sk}(\tilde{\mathbf{u}})R = R\mathrm{Sk}(\mathbf{v})$$

2. Rotations in three dimensions.

Consider any matrix $Q \in SO(3)$.

a) Show that all the eigenvalues of Q are on the unit circle in the complex plane.

b) Show that Q always has an eigenvalue of unity, and so there is a unit vector $\mathbf{w}$ such that $Q\mathbf{w} = \mathbf{w}$. This vector defines the axis of rotation of Q and is parallel to the axial vector of the skew matrix $Q - Q^T$. Can a proper rotation have more than one axis?

c) Let $\mathbf{v}$ be any unit vector orthogonal to $\mathbf{w}$. Show that $Q\mathbf{v}$ is also a unit vector orthogonal to $\mathbf{w}$ and that the angle $0 \le \theta \le \pi$ between $\mathbf{v}$ and $Q\mathbf{v}$ satisfies the relation

$$1 + 2\cos\theta = \mathrm{tr}(Q). \tag{1}$$

[Hint: Express $\mathrm{tr}(Q)$ in terms of the eigenvalues.]

d) Given a unit vector $\mathbf{n}$ along the axis of a right-handed rotation of angle ϕ, the matrix $Q \in SO(3)$ associated with the rotation in the basis $\{\mathbf{e}_1, \mathbf{e}_2, \mathbf{e}_3\}$ is given by

$$Q = \cos\phi\,\mathrm{Id} + (1 - \cos\phi)\,\mathbf{n} \otimes \mathbf{n} + \sin\phi\,\mathbf{n}^{\times}. \tag{2}$$

Show that (2) can also be expressed as

$$Q = \mathrm{Id} + \sin\phi\,\mathbf{n}^{\times} + (1 - \cos\phi)\,\mathbf{n}^{\times}\,\mathbf{n}^{\times}. \tag{3}$$

[Hint: First distribute the triple product $\mathbf{n} \times (\mathbf{n} \times \mathbf{v})$ for arbitrary vector $\mathbf{v}$.]

3. General form of the Darboux vector of an adapted framing of a given curve.

Given a smooth curve $\mathbf{r}(s)$ and a function $\mathsf{u}_3(s)$, where s is the arclength parameter, we will show that

$$\boldsymbol{\xi}' = (\mathsf{u}_3\,\mathbf{r}' + \mathbf{r}' \times \mathbf{r}'') \times \boldsymbol{\xi} \tag{4}$$

with initial condition

$$\boldsymbol{\xi}(0) \cdot \mathbf{r}'(0) = 0, \quad |\boldsymbol{\xi}(0)|^2 = 1, \tag{5}$$

generates an orthonormal framing $(\boldsymbol{\xi}, (\mathbf{r}' \times \boldsymbol{\xi}), \mathbf{r}')$ of $\mathbf{r}(s)$.
Verify and calculate

a) That $|\boldsymbol{\xi}(0)|^2 = 1 \implies |\boldsymbol{\xi}(s)|^2 = 1 \quad \forall s$.

b) That $\boldsymbol{\xi}(0) \cdot \mathbf{r}'(0) = 0 \implies \boldsymbol{\xi}(s) \cdot \mathbf{r}'(s) = 0 \quad \forall s$.

c) Now, by picking an initial value of $\boldsymbol{\xi}(0)$ satisfying (5) we have an orthonormal frame $(\boldsymbol{\xi}, (\mathbf{r}' \times \boldsymbol{\xi}), \mathbf{r}')$ of $\mathbf{r}(s)$. What is the Darboux vector

d) Note that $\mathbf{y} = \mathbf{r}'$ and $\mathbf{z} = \mathbf{r}' \times \boldsymbol{\xi}$ must be two other solutions of (4). Check!

e) If $\mathbf{r}'' \neq 0$, what are the components of the Darboux vector in the Serret-Frenet frame?

f) If $\mathbf{r} \in C^3$ and $\mathbf{r}''(s) \neq 0$ for all s, show that the principal normal $\mathbf{n}$ to $\mathbf{r}$ solves (4) when $\mathsf{u}_3 = \tau$, where τ is the torsion of $\mathbf{r}$.

The facts (a) and (b) say that $|\boldsymbol{\xi}|^2$ and $\boldsymbol{\xi} \cdot \mathbf{r}'$ are integrals of the system (4).

4. Frenet-Serret equations in $\mathbb{R}^n$. (optional, not examinable)

Given a curve $\mathbf{r} : \mathbb{R} \mapsto \mathbb{R}^n$ parameterised by arc-length and such that the n vectors $\{\mathbf{r}', \mathbf{r}'', \cdots, \mathbf{r}^{(n)}\}$ are linearly independent, prove that there exists an *orthogonal* basis of $\mathbb{R}^n$ $\{\mathbf{t} = \mathbf{r}', \mathbf{n}_1, \cdots, \mathbf{n}_{n-1}\}$ such that

$$\begin{pmatrix}\mathbf{t} & \mathbf{n}_1 & \cdots & \mathbf{n}_n\end{pmatrix}' = \begin{pmatrix}\mathbf{t} & \mathbf{n}_1 & \cdots & \mathbf{n}_n\end{pmatrix} \begin{pmatrix} 0 & -\kappa_1 & 0 & \dots & 0 & 0 & 0 \\ \kappa_1 & 0 & -\kappa_2 & \dots & 0 & 0 & 0 \\ 0 & \kappa_2 & 0 & \ddots & 0 & 0 & 0 \\ \vdots & \vdots & \ddots & \ddots & \ddots & 0 & 0 \\ 0 & 0 & 0 & \ddots & 0 & -\kappa_{n-2} & 0 \\ 0 & 0 & 0 & \dots & \kappa_{n-2} & 0 & -\kappa_{n-1} \\ 0 & 0 & 0 & \dots & 0 & \kappa_{n-1} & 0 \end{pmatrix}. \tag{6}$$

In this lecture, we prefer to have the tangent to the curve as the last entry, how does equation (6) adapt if we consider the basis $(\mathbf{n}_1 \; \mathbf{n}_2 \; \cdots \; \mathbf{n}_{n-1} \; \mathbf{t})$?