1. Given an orthonormal basis \( \{ e_1, e_2, e_3 \} \) and a particular vector \( u \) in \( \mathbb{R}^3 \), denote by \( \text{Sk}(u) \) the operator that gives \( \text{Sk}(u)v = u \wedge v \). Show that for any invertible matrix \( M \in \mathbb{R}^3 \times \mathbb{R}^3 \),
\[
\text{Sk}(Mu) = |M| M^{-T} \text{Sk}(u) M^{-1}.
\]
Show that if \( M \in SO(3) \) then this formula simplifies to
\[
\text{Sk}(Mu) = MSk(u)M^{-1}
\]
Now let \( \tilde{u} \) be the Darboux vector of a curve \( R \in SO(3) \) so \( d'_i = \tilde{u} \wedge d_i \) where \( d_i \) are the columns of \( R \) and let \( v \) be a vector such that \( \tilde{u} = Rv \). Show that
\[
R' = \text{Sk}(\tilde{u})R = RSk(v)
\]

2. Rotations in three dimensions.
Consider any matrix \( Q \in SO(3) \).

a) Show that all the eigenvalues of \( Q \) are on the unit circle in the complex plane.

b) Show that \( Q \) always has an eigenvalue of unity, and so there is a unit vector \( w \) such that \( Qw = w \). This vector defines the axis of rotation of \( Q \) and is parallel to the axial vector of the skew matrix \( Q - Q^T \). Can a proper rotation have more than one axis?

c) Let \( v \) be any unit vector orthogonal to \( w \). Show that \( Qv \) is also a unit vector orthogonal to \( w \) and that the angle \( 0 \leq \theta \leq \pi \) between \( v \) and \( Qv \) satisfies the relation
\[
1 + 2 \cos \theta = \text{tr}(Q).
\]
[Hint: Express \( \text{tr}(Q) \) in terms of the eigenvalues.]

d) Given a unit vector \( n \) along the axis of a right-handed rotation of angle \( \phi \), the matrix \( Q \in SO(3) \) associated with the rotation in the basis \( \{ e_1, e_2, e_3 \} \) is given by
\[
Q = \cos \phi \text{Id} + (1 - \cos \phi) n \otimes n + \sin \phi n^x.
\]
Show that (2) can also be expressed as
\[
Q = \text{Id} + \sin \phi n^x + (1 - \cos \phi) n^x n^x.
\]
[Hint: First distribute the triple product \( n \times (n \times v) \) for arbitrary vector \( v \).]

Given a smooth curve \( r(s) \) and a function \( u_3(s) \), where \( s \) is the arclength parameter, we will show that
\[
\xi' = (u_3 r' + r' \times r'') \times \xi
\]
with initial condition
\[
\xi(0) \cdot r'(0) = 0, \quad |\xi(0)|^2 = 1,
\]
generates an orthonormal framing \( (\xi, (r' \times \xi), r') \) of \( r(s) \).
Verify and calculate
a) That $|\xi(0)|^2 = 1 \implies |\xi(s)|^2 = 1 \ \forall s$.

b) That $\xi(0) \cdot r'(0) = 0 \implies \xi(s) \cdot r'(s) = 0 \ \forall s$.

c) Now, by picking an initial value of $\xi(0)$ satisfying (5) we have an orthonormal frame $(\xi, (r' \times \xi), r')$ of $r(s)$. What is the Darboux vector

d) Note that $y = r'$ and $z = r' \times \xi$ must be two other solutions of (4). Check!

e) If $r'' \neq 0$, what are the components of the Darboux vector in the Serret-Frenet frame?

f) If $r \in C^3$ and $r''(s) \neq 0$ for all $s$, show that the principal normal $n$ to $r$ solves (4) when $u_3 = \tau$, where $\tau$ is the torsion of $r$.

The facts (a) and (b) say that $|\xi|^2$ and $\xi \cdot r'$ are integrals of the system (4).

4. Frenet-Serret equations in $\mathbb{R}^n$. (optional, not examinable)

Given a curve $r : \mathbb{R} \rightarrow \mathbb{R}^n$ parameterised by arc-length and such that the $n$ vectors $\{r', r'', \ldots, r^{(n)}\}$ are linearly independent, prove that there exists an orthogonal basis of $\mathbb{R}^n \{t = r', n_1, \ldots, n_{n-1}\}$ such that

$$
\begin{pmatrix}
0 & -\kappa_1 & 0 & \ldots & 0 & 0 & 0 \\
\kappa_1 & 0 & -\kappa_2 & \ldots & 0 & 0 & 0 \\
0 & \kappa_2 & 0 & \ldots & 0 & 0 & 0 \\
\vdots & \vdots & \vdots & \ddots & \vdots & \vdots & \vdots \\
0 & 0 & 0 & \ldots & 0 & -\kappa_{n-2} & 0 \\
0 & 0 & 0 & \ldots & \kappa_{n-2} & 0 & -\kappa_{n-1} \\
0 & 0 & 0 & \ldots & 0 & \kappa_{n-1} & 0
\end{pmatrix}
$$

(6)

In this lecture, we prefer to have the tangent to the curve as the last entry, how does equation (6) adapt if we consider the basis $(n_1, n_2, \ldots, n_{n-1}, t)$?