In this exercise session, $t$ stands for a generic parameterisation of curves. In particular, it is not necessarily the arc-length.

1. Composition of Darboux vectors

Given a curve $\mathbf{r}(t)$ and two orthonormal framings $\{\mathbf{d}_i(t)\}$ and $\{\mathbf{D}_i(t)\}$ where $\mathbf{D}_i$ and $\mathbf{d}_i$ are column vectors, their two direction cosine matrices $d$ and $D$ are related by a rotation matrix $Q \in SO(3)$

$$D = dQ,$$

where each matrix can depend upon $t$. The relation (1) implies that $Q_{ij} = \mathbf{d}_i \cdot \mathbf{D}_j$.

Let $\mathbf{u}(t) = u_i(t) \mathbf{d}_i(t)$ be the Darboux vector associated with the frame $\{\mathbf{d}_i\}$ with components $u$ satisfying

$$\mathbf{u} = d^T \mathbf{d},$$

and let $\mathbf{U}(t) = U_i(t) \mathbf{D}_i(t)$ be the Darboux vector associated with the frame $\{\mathbf{D}_i\}$ with components $U$ satisfying

$$\mathbf{U} = D^T \mathbf{D}'.$$

a) Show that

$$\mathbf{U} = Q^T \mathbf{u} + Q^T \mathbf{Q}'.$$

b) Accordingly, show that

$$\mathbf{U} = \mathbf{u} + \mathbf{D}_i \mathbf{p}_i,$$

where the components $\mathbf{p}$ respect $\mathbf{p} = Q^T \mathbf{Q}'.$

c) Simplify (2) for the case when the two frames $\{\mathbf{D}_i\}$ and $\{\mathbf{d}_i\}$ share a common vector $\mathbf{D}_3(t) = \mathbf{d}_3(t)$, as would arise in the case of any two adapted framings.

d) In the further case when $\{\mathbf{D}_i\}$ is the Frenet frame, so that $\mathbf{D}_3 = \mathbf{t} = \mathbf{d}_3$, find the explicit expressions of the components of $\mathbf{u}$ in the Frenet frame in function of the curvature $\kappa$ and torsion $\tau$ of the curve $\mathbf{r}$.

2. Factorisation of curves in $SE(3)$

Let $\mathcal{X}(t)$ and $\mathcal{Y}(t)$ be two curves in $SE(3)$ with homogeneous coordinates

$$\mathcal{X}(t) = \begin{pmatrix} \mathbf{X} & \mathbf{x} \\ 0 & 1 \end{pmatrix}, \quad \text{and} \quad \mathcal{Y}(t) = \begin{pmatrix} \mathbf{Y} & \mathbf{y} \\ 0 & 1 \end{pmatrix}.$$

Define a third curve in $SE(3)$ via

$$\mathcal{Z}(t) = \mathcal{X}(t) \mathcal{Y}(t).$$

There exist vectors $\mathbf{u}$, $\mathbf{v}$, $\mathbf{w}$, $\mathbf{a}$, $\mathbf{b}$ and $\mathbf{c}$ such that

$$\mathcal{X}' = \mathcal{X} \begin{pmatrix} \mathbf{u} & \mathbf{a} \\ 0 & 0 \end{pmatrix}, \quad \mathcal{Y}' = \mathcal{Y} \begin{pmatrix} \mathbf{v} & \mathbf{b} \\ 0 & 0 \end{pmatrix}, \quad \text{and} \quad \mathcal{Z}' = \mathcal{Z} \begin{pmatrix} \mathbf{w} & \mathbf{c} \\ 0 & 0 \end{pmatrix}.$$

In parallel to what you have already seen for curves in $SO(3)$, find an expression for $\mathbf{w}$ and $\mathbf{c}$ as a function of $\mathcal{X}$, $\mathcal{Y}$, $\mathbf{u}$, $\mathbf{v}$, $\mathbf{a}$ and $\mathbf{b}$.
3. Offset of a curve in $\mathbb{R}^3$

Assume that
\[
\mathcal{X}(t) = \begin{pmatrix} X(t) & x(t) \\ 0 & 1 \end{pmatrix},
\]
is the $SE(3)$ curve corresponding to $x(t)$, a prescribed curve in $\mathbb{R}^3$ equipped with the adapted frame $X(t) = [d_1 \ d_2 \ d_3]$ where $t$ is not necessarily arc-length.

Given a real number $\epsilon > 0$, the curve
\[
z(t) = x(t) + \epsilon d_1(t),
\]
in $\mathbb{R}^3$ is called an offset of $x$.

a) Under what condition can you guarantee that $z'(t) \neq 0$ for all $t$?

b) Show that if $z'(t) \neq 0$ for all $t$, it is possible to equip the $\mathbb{R}^3$ curve $z(t)$ with an adapted frame
\[
\mathcal{Z}(t) = \begin{pmatrix} D(t) & z(t) \\ 0 & 1 \end{pmatrix},
\]
where $D(t) = (D_1(t) \ D_2(t) \ D_3(t))$ and such that $D_1(t) = d_1(t)$.

c) Find the curve $\mathcal{Y}(t) \in SE(3)$ such that
\[
\mathcal{Z}(t) = \mathcal{X}(t)\mathcal{Y}(t).
\]

Compute the explicit form of $w$ and $c$ from exercise 2.

[ hint: $Y$ is of the form $Y = \begin{pmatrix} 1 & 0 & 0 \\ 0 & \cos \varphi & -\sin \varphi \\ 0 & \sin \varphi & \cos \varphi \end{pmatrix}$. ]