In this exercise session, \( t \) stands for a generic parameterisation of curves. In particular, it is not necessarily the arc-length.

1. Composition of Darboux vectors

Given a curve \( \mathbf{r}(t) \) and two orthonormal framings \( \{\mathbf{d}_i(t)\} \) and \( \{\mathbf{D}_i(t)\} \) where \( \mathbf{D}_i \) and \( \mathbf{d}_i \) are column vectors, their two direction cosine matrices \( d \) and \( D \) are related by a rotation matrix \( Q \in SO(3) \)

\[
D = d Q,
\]

where each matrix can depend upon \( t \). The relation (1) implies that \( Q_{ij} = \mathbf{D}_i \cdot \mathbf{d}_j \).

Let \( \mathbf{u}(t) = u_i(t) \mathbf{d}_i(t) \) be the Darboux vector associated with the frame \( \{\mathbf{d}_i\} \) with components \( u \) satisfying

\[
\mathbf{u}^\times = d^T d',
\]

and let \( \mathbf{U}(t) = U_i(t) \mathbf{D}_i(t) \) be the Darboux vector associated with the frame \( \{\mathbf{D}_i\} \) with components \( U \) satisfying

\[
\mathbf{U}^\times = D^T D'.
\]

a) Show that

\[
\mathbf{U}^\times = Q^T \mathbf{u}^\times Q + Q^T Q'.
\]

b) Accordingly, show that

\[
\mathbf{U} = \mathbf{u} + \mathbf{D}_i \mathbf{p}_i,
\]

where the components \( \mathbf{p} \) respect \( \mathbf{p}^\times = Q^T Q' \).

(c) Simplify (2) for the case when the two frames \( \{\mathbf{D}_i\} \) and \( \{\mathbf{d}_i\} \) share a common vector \( \mathbf{D}_3(t) = \mathbf{d}_3(t) \), as would arise in the case of any two adapted framings.

d) In the further case when \( \{\mathbf{D}_i\} \) is the Frenet frame, so that \( \mathbf{D}_3 = \mathbf{t} = \mathbf{d}_3 \), find the explicit expressions of the components of \( \mathbf{u} \) in the Frenet frame in function of the curvature \( \kappa \) and torsion \( \tau \) of the curve \( \mathbf{r} \).

2. Factorisation of curves in \( SE(3) \)

Let \( \mathbf{X}(t) \) and \( \mathbf{Y}(t) \) be two curves in \( SE(3) \) with homogeneous coordinates

\[
\mathbf{X}(t) = \begin{pmatrix} \mathbf{X} & \mathbf{x} \\ 0 & 1 \end{pmatrix}, \quad \text{and} \quad \mathbf{Y}(t) = \begin{pmatrix} \mathbf{Y} & \mathbf{y} \\ 0 & 1 \end{pmatrix}.
\]

Define a third curve in \( SE(3) \) via

\[
\mathbf{Z}(t) = \mathbf{X}(t) \mathbf{Y}(t).
\]

There exist vectors \( \mathbf{u}, \mathbf{v}, \mathbf{w}, \mathbf{a}, \mathbf{b} \) and \( \mathbf{c} \) such that

\[
\mathbf{X}' = \mathbf{X} \begin{pmatrix} \mathbf{u}^\times & \mathbf{a} \\ 0 & 0 \end{pmatrix}, \quad \mathbf{Y}' = \mathbf{Y} \begin{pmatrix} \mathbf{v}^\times & \mathbf{b} \\ 0 & 0 \end{pmatrix}, \quad \text{and} \quad \mathbf{Z}' = \mathbf{Z} \begin{pmatrix} \mathbf{w}^\times & \mathbf{c} \\ 0 & 0 \end{pmatrix}.
\]

In parallel to what you have already seen for curves in \( SO(3) \), find an expression for \( \mathbf{w} \) and \( \mathbf{c} \) as a function of \( \mathbf{X}', \mathbf{Y}', \mathbf{u}, \mathbf{v}, \mathbf{a} \) and \( \mathbf{b} \).
3. Offset of a curve in $\mathbb{R}^3$

Assume that

$$X(t) = \begin{pmatrix} X(t) & x(t) \\ 0 & 1 \end{pmatrix},$$

is the $SE(3)$ curve corresponding to $x(t)$, a prescribed curve in $\mathbb{R}^3$ equipped with the adapted frame $X(t) = [d_1 \ d_2 \ d_3]$ where $t$ is not necessarily arc-length.

Given a real number $\epsilon > 0$, the curve

$$z(t) = x(t) + \epsilon d_1(t),$$

in $\mathbb{R}^3$ is called an offset of $x$.

a) Under what condition can you guarantee that $z'(t) \neq 0$ for all $t$?

b) Show that if $z'(t) \neq 0$ for all $t$, it is possible to equip the $\mathbb{R}^3$ curve $z(t)$ with an adapted frame

$$Z(t) = \begin{pmatrix} D(t) & z(t) \\ 0 & 1 \end{pmatrix},$$

where $D(t) = (D_1(t) \ D_2(t) \ D_3(t))$ and such that $D_1(t) = d_1(t)$.

c) Find the curve $Y(t) \in SE(3)$ such that

$$Z(t) = X(t)Y(t).$$

Compute the explicit form of $w$ and $c$ from exercise 1.

[hint: $Y$ is of the form $Y = \begin{pmatrix} 1 & 0 & 0 \\ 0 & \cos \varphi & -\sin \varphi \\ 0 & \sin \varphi & \cos \varphi \end{pmatrix}$]