Given two smooth closed oriented and smoothly closed curves \( C_1 \) (or \( x(s) \)) and \( C_2 \) (or \( y(\sigma) \)) in \( \mathbb{R}^3 \) such that \( C_1 \cap C_2 = \{ \emptyset \} \) (that is there no intersection between \( C_1 \) and \( C_2 \)), we defined their Linking number \( \text{Lk} \) as

\[
\text{Lk}(C_1,C_2) = \frac{1}{4\pi} \int_{C_1} \int_{C_2} \frac{(y(\sigma) - x(s)) \cdot (y'(\sigma) \times x'(s))}{||y(\sigma) - x(s)||^3} d\sigma ds.
\]

Here \( \sigma \) and \( s \) are not necessarily arc-length parameterisations. As shown in class the value of \( \text{Lk} \) is unaffected by orientation-preserving reparametrisations of either curve; and the sign of \( \text{Lk} \) is switched if the orientation of one curve is switched.

Note that while the non-intersection of \( y(\sigma) \) and \( x(s) \) is an important hypothesis, the self-intersection of \( y(\sigma) \) with itself, or \( x(s) \) with itself, is not a significant difficulty.

1 Link as a signed area

Introduce the unit vector field \( e(\sigma, s) = \frac{y(\sigma) - x(s)}{||y(\sigma) - x(s)||} \) and notice that because the curves \( y \) and \( x \) do not intersect, \( e(\sigma, s) \) is well-defined and smooth for all \( \sigma \) and \( s \). It is also periodic because \( y(\sigma) \) and \( x(s) \) are periodic.

If the triple bracket \([a, b, c]\) denotes the scalar triple product \([a, b, c] = a \cdot (b \times c)\), show that

\[
[e, e_s, e_\sigma] = \frac{(y(\sigma) - x(s)) \cdot (y'(\sigma) \times x'(s))}{||y(\sigma) - x(s)||^3},
\]

where \( e_\sigma \) and \( e_s \) are partial derivatives of \( e(\sigma, s) \) (at fixed \( y \) and \( x \)).

Accordingly, the Link integral can be rewritten as:

\[
\frac{1}{4\pi} \int_{C_1} \int_{C_2} [e, e_s, e_\sigma] d\sigma ds.
\]

Show that therefore, \( \text{Lk} \) is the signed surface area of a (often multi-covered) portion of a sphere. We will discuss this further in next week’s lecture.

2 Homotopy invariance

First order variations \( x(s; \epsilon) = x(s) + \epsilon \delta x(s) \) and \( y(\sigma; \epsilon) = y(\sigma) + \epsilon \delta y(\sigma) \) of the curves \( x \) and \( y \) generate a first order variation \( e(s, \sigma; \epsilon) = e(s, \sigma) + \epsilon \delta e(s, \sigma) \) of the unit vector field \( e(s, \sigma) \). Compute

\[
\delta e = \frac{de}{d\epsilon} \bigg|_{\epsilon=0}.
\]

Then show that the corresponding variation of the Link integral, i.e.

\[
\delta \text{Lk} = \frac{d}{d\epsilon} \text{Lk}(x + \epsilon \delta x, y + \epsilon \delta y) \bigg|_{\epsilon=0},
\]

is identically zero for all doubly-periodic unit vector fields \( e \).

[Hints: Notice that any derivative of \([a, b, c]\) satisfies the ‘product rule’: \([a, b, c]' = [a', b, c] + [a, b', c] + [a, b, c']\). You must integrate by parts (in \( s \) on the \( \delta e_s \) term and in \( \sigma \) on the \( \delta e_\sigma \) term) and use periodicity to conclude that no boundary term arises. You must also use skew-symmetry properties of the triple...
product, and the fact that because \( e(\sigma, s) \) is a unit vector field \( e \cdot e_s = e \cdot e_\sigma = e \cdot \delta e = 0 \), or in other words \( e_s, e_\sigma \) and \( \delta e \) are co-planar so that \( [e_s, e_\sigma, \delta e] = 0 \).

This computation is valid at all non-intersecting \( y \) and \( x \) (so that \( e \) is smooth), and suffices to show that \( Lk \) is a homotopy invariant.

3 The Hopf link

The Hopf link is represented on the left of the following figure.

Evaluate its Link integral by explicit integration after choosing explicit parameterizations where the second loop \( y(\sigma) \) is formed by a straight segment \([−L, L]\) of the \( z \)-axis plus a smooth closure lying outside the ball of radius \( L \) and the unit circle \( x(s) \) is in the \( xy \) plane. You need to show

1. that the contribution to the \( Lk \) integral from the part of the curve \( y(\sigma) \) outside the ball of radius \( L \) is arbitrarily small for \( L \rightarrow \infty \) (which is a homotopy under which \( Lk \) is invariant).

2. that the remaining part of the integral concerning the Link between the straight part of \( y \) and the circle \( x \) is an integer. What is the interpretation of this integer?
   (Hint: Use substitution by \( \sinh z \) and the fact that \( \tanh' z = \frac{1}{\cosh^2 z} \) and \( \tanh z \rightarrow 1 \) for \( z \rightarrow \infty \).)

How would you use today’s results to compute \( Lk \) for the link on the right of the figure?