Differential Geometry of Framed Curves
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Session 4: Exercises T. Lessinnes

Given two smooth closed oriented and smoothly closed curves C_1 (or $\mathbf{x}(s)$) and C_2 (or $\mathbf{y}(\sigma)$) in $\mathbb{R}^3$ such that $C_1 \cap C_2 = \{\varnothing\}$ (that is there no intersection between C_1 and C_2), we defined their Linking number Lk as

$$\mathrm{Lk}(C_1, C_2) = \frac{1}{4\pi}\int_{C_1}\int_{C_2} \frac{(\mathbf{y}(\sigma)-\mathbf{x}(s))\cdot(\mathbf{y}'(\sigma)\times\mathbf{x}'(s))}{||\mathbf{y}(\sigma)-\mathbf{x}(s)||^3} d\sigma\, ds.$$

Here σ and s are not necessarily arc-length parameterisations. As shown in class the value of Lk is unaffected by orientation-preserving reparametrisations of either curve; and the sign of Lk is switched if the orientation of one curve is switched.

Note that while the non-intersection of $\mathbf{y}(\sigma)$ and $\mathbf{x}(s)$ is an important hypothesis, the self-intersection of $\mathbf{y}(\sigma)$ with itself, or $\mathbf{x}(s)$ with itself, is not a significant difficulty.

1 Link as a signed area

Introduce the unit vector field $\mathbf{e}(\sigma, s) = \frac{\mathbf{y}(\sigma)-\mathbf{x}(s)}{||\mathbf{y}(\sigma)-\mathbf{x}(s)||}$ and notice that because the curves $\mathbf{y}$ and $\mathbf{x}$ do not intersect, $\mathbf{e}(\sigma, s)$ is well-defined and smooth for all σ and s. It is also periodic because $\mathbf{y}(\sigma)$ and $\mathbf{x}(s)$ are periodic.

If the triple bracket $[\mathbf{a}, \mathbf{b}, \mathbf{c}]$ denotes the scalar triple product $[\mathbf{a}, \mathbf{b}, \mathbf{c}] = \mathbf{a}\cdot(\mathbf{b}\times\mathbf{c})$, show that

$$[\mathbf{e}, \mathbf{e}_s, \mathbf{e}_\sigma] = \frac{(\mathbf{y}(\sigma)-\mathbf{x}(s))\cdot(\mathbf{y}'(\sigma)\times\mathbf{x}'(s))}{||\mathbf{y}(\sigma)-\mathbf{x}(s)||^3},$$

where $\mathbf{e}_\sigma$ and $\mathbf{e}_s$ are partial derivatives of $\mathbf{e}(\sigma, s)$ (at fixed $\mathbf{y}$ and $\mathbf{x}$).

Accordingly, the Link integral can be rewritten as:

$$\frac{1}{4\pi}\int_{C_1}\int_{C_2} [\mathbf{e}, \mathbf{e}_s, \mathbf{e}_\sigma]\, d\sigma\, ds. \tag{1}$$

Show that therefore, Lk is the signed surface area of a (often multi-covered) portion of a sphere. We will discuss this further in next week's lecture.

2 Homotopy invariance

First order variations $\mathbf{x}(s;\epsilon) = \mathbf{x}(s) + \epsilon\,\delta\mathbf{x}(s)$ and $\mathbf{y}(\sigma;\epsilon) = \mathbf{y}(\sigma) + \epsilon\,\delta\mathbf{y}(\sigma)$ of the curves $\mathbf{x}$ and $\mathbf{y}$ generate a first order variation $\mathbf{e}(s,\sigma;\epsilon) = \mathbf{e}(s,\sigma) + \epsilon\,\delta\mathbf{e}(s,\sigma)$ of the unit vector field $\mathbf{e}(s,\sigma)$. Compute

$$\delta\mathbf{e} = \left.\frac{\mathrm{d}\mathbf{e}}{\mathrm{d}\epsilon}\right|_{\epsilon=0}.$$

Then show that the corresponding variation of the Link integral, i.e.

$$\delta\mathrm{Lk} = \left.\frac{\mathrm{d}}{\mathrm{d}\epsilon}\mathrm{Lk}(\mathbf{x}+\epsilon\,\delta\mathbf{x}, \mathbf{y}+\epsilon\,\delta\mathbf{y})\right|_{\epsilon=0},$$

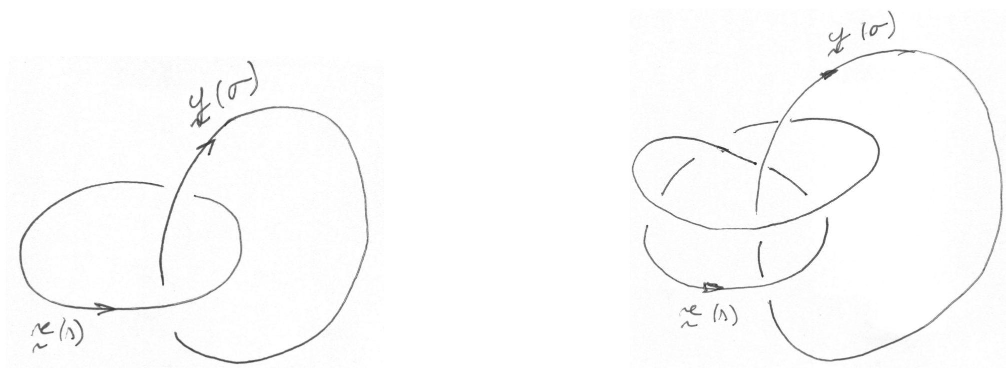
is identically zero for all doubly-periodic unit vector fields $\mathbf{e}$.
[Hints: Notice that any derivative of $[\mathbf{a}, \mathbf{b}, \mathbf{c}]$ satisfies the 'product rule': $[\mathbf{a}, \mathbf{b}, \mathbf{c}]' = [\mathbf{a}', \mathbf{b}, \mathbf{c}] + [\mathbf{a}, \mathbf{b}', \mathbf{c}] + [\mathbf{a}, \mathbf{b}, \mathbf{c}']$. You must integrate by parts (in s on the $\delta\mathbf{e}_s$ and in σ on the $\delta\mathbf{e}_\sigma$ term) and use periodicity to conclude that no boundary term arises. You must also use skew-symmetry properties of the triple

product, and the fact that because $\mathbf{e}(\sigma, s)$ is a unit vector field $\mathbf{e} \cdot \mathbf{e}_s = \mathbf{e} \cdot \mathbf{e}_\sigma = \mathbf{e} \cdot \delta\mathbf{e} = 0$, or in other words $\mathbf{e}_s$, $\mathbf{e}_\sigma$ and $\delta\mathbf{e}$ are co-planar so that $[\mathbf{e}_s, \mathbf{e}_\sigma, \delta\mathbf{e}] = 0$.]

This computation is valid at all non-intersecting $\mathbf{y}$ and $\mathbf{x}$ (so that $\mathbf{e}$ is smooth), and suffices to show that Lk is a homotopy invariant.

3 The Hopf link

The Hopf link is represented on the left of the following figure.



Evaluate its Link integral by explicit integration after choosing explicit parameterizations where the second loop $\mathbf{y}(\sigma)$ is formed by a straight segment $[-L, L]$ of the z-axis plus a smooth closure lying outside the ball of radius L and the unit circle $\mathbf{x}(s)$ is in the xy plane. You need to show

1. that the contribution to the Lk integral from the part of the curve $y(\sigma)$ outside the ball of radius L is arbitrarily small for $L \to \infty$ (which is a homotopy under which Lk is invariant).

2. that the remaining part of the integral concerning the Link between the straight part of $\mathbf{y}$ and the circle $\mathbf{x}$ is an integer. What is the interpretation of this integer?
(Hint: Use substitution by $\sinh z$ and the fact that $\tanh' z = \frac{1}{\cosh^2 z}$ and $\tanh z \to 1$ for $z \to \infty$.)

How would you use today's results to compute Lk for the link on the right of the figure?