Differential Geometry of Framed Curves
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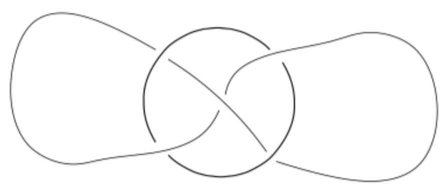
Session 5: Exercises T. Lessinnes

1 Unknotted curves and the Whitehead link

If $\mathbf{y}(\sigma)$ and $\mathbf{x}(s)$ can be moved by homotopy, while keeping $C_1 \cap C_2 = \emptyset$, to opposite sides of a plane (in particular if $\mathbf{y}$ and $\mathbf{x}$ are not linked to each other in an intuitive sense), then $\mathrm{Lk}(\mathbf{y}, \mathbf{x}) = 0$. This is obviously true if we consider the definition of Lk based on signed crossings.

To prove this from the point of view of the integral definition, show that Lk can be bounded from above by an arbitrarily small number by the homotopy in which $\mathbf{x}(s)$ (say) is moved to infinity by translation along the normal to the separating plane.

Unfortunately the converse is not true. Use the method of counting signed crossings to show that the Whitehead link shown in the figure below, and for any choices of orientations, has Lk $= 0$ despite the fact that the two strands are physically linked i.e. they cannot be moved arbitrarily far apart under a homotopy respecting $\mathbf{x}_\epsilon(s_1) \neq \mathbf{x}_\epsilon(s_2)\ \forall s_1 \neq s_2$, and $\mathbf{y}_\epsilon(\sigma_1) \neq \mathbf{y}_\epsilon(\sigma_2)\ \forall \sigma_1 \neq \sigma_2$. How could you use the



result of Problem 2, week 4 to show (without counting signed crossings) that the Whitehead link integral vanishes?

For the Whitehead link Lk=0, as by crossing one curve with itself, you can 'physically unlink' the two curves. As seen in Problem 1, week 4 this deformation doesn't change the link.

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Given two smooth closed oriented curves $\mathbf{x}(s)$ and $\mathbf{y}(\sigma)$ parameterised by their respective arc-length and such that $\mathbf{x}''(s)$ and $\mathbf{y}''(\sigma)$ never vanish, consider the surface $\mathbf{r}(s, \sigma) = \mathbf{y}(\sigma) - \mathbf{x}(s)$. Whenever $\mathbf{r}_s \wedge \mathbf{r}_\sigma \neq 0$, the unit normal $\mathbf{n}$ to $\mathbf{r}$ is defined by

$$\mathbf{n}(s, \sigma) = \frac{\mathbf{r}_s \wedge \mathbf{r}_\sigma}{||\mathbf{r}_s \wedge \mathbf{r}_\sigma||}. \tag{1}$$

1. **Taylor expansion warm up.** Compute the Taylor expansion of $\mathbf{r}$ around a prescribed point (s_0, σ_0) up to second order. Then, express your result as a function of the Frenet frames and curvatures of $\mathbf{x}$ and $\mathbf{y}$. Also compute the Taylor expansion of $\mathbf{r}_s \wedge \mathbf{r}_\sigma$ to first order.

2. **In general, it is not possible to define a continuous field of unit normals on r.** Prove that if there exists a point (s_0, σ_0) such that the tangent to $\mathbf{x}$ at s_0 is parallel to the tangent to $\mathbf{y}$ at σ_0, then it is impossible to complete the definition (1) of $\mathbf{n}$ in a continuous way. What do you think $\mathbf{r}$ looks like close to such a point?

3. **Such pathological points are isolated provided that $\mathbf{x}_{ss}(s_0) \wedge \mathbf{y}_{\sigma\sigma}(\sigma_0) \neq 0$.** Given a point (s_0, σ_0) such that the tangent to $\mathbf{x}$ at s_0 is parallel to the tangent to $\mathbf{y}$ at σ_0 and $\mathbf{x}_{ss}(s_0) \wedge \mathbf{y}_{\sigma\sigma}(\sigma_0) \neq 0$, prove that it is not possible to choose a positive number $a > 0$ and a regular curve

$$\gamma : t \in (-a, a) \to \big(s(t), \sigma(t)\big),$$

such that both $\gamma(0) = (s_0, \sigma_0)$ and $\mathbf{r}_s\big(\gamma(t)\big) \wedge \mathbf{r}_\sigma\big(\gamma(t)\big) = 0$ for all $t \in (-a, a)$.

4. **Provided that the two curves x and y never share an osculating plane**[1]**, points subtending tangent rays from the origin**[2] **form smooth curves on r.** Define the function $f(s, \sigma) = \mathbf{r}(s, \sigma) \cdot \big(\mathbf{r}_s(s, \sigma) \wedge \mathbf{r}_\sigma(s, \sigma)\big)$. Show that if $f(s_0, \sigma_0) = 0$, then there exists an (unique) open curve γ such that (s_0, σ_0) is in the image of γ and $f(\gamma) = 0$.

5. **Provided that the two curves x and y do not share an osculating plane, the pathological points are contained in curves on r subtending tangent rays from the origin.** Given a point (s_0, σ_0) such that the tangent to $\mathbf{x}$ at s_0 is parallel to the tangent to $\mathbf{y}$ at σ_0 and $\mathbf{x}_{ss}(s_0) \wedge \mathbf{y}_{\sigma\sigma}(\sigma_0) \neq 0$, show that there exists a curve $\gamma(t) = \big(s(t), \sigma(t)\big)$ and a positive number a such that $f\big(\gamma(t)\big) = 0$ for all $t \in (-a, a)$.

[1] The osculating plane is the plane spanned by the tangent vector and the normal vector.

[2] These are points $\mathbf{p}$ on the surface $\mathbf{r}$ such that the ray from the origin through $\mathbf{p}$ is tangent to the surface $\mathbf{r}$.