Differential Geometry of Framed Curves
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SESSION 6: EXERCISES T. LESSINNES

1 Curves on a sphere

1.1 A general curve lying on a sphere

Consider a curve $\mathbf{w}(\sigma)$ lying on the sphere $\partial B(\mathbf{0}, R)$ of radius R centred at the origin of $\mathbb{R}^3$. Assume that σ is arc-length in $\mathbf{w}$: $\mathbf{w}'(\sigma) = \mathbf{v}(\sigma)$ where $\mathbf{v}$ is a unit vector. Also consider $\mathbf{N}(\sigma)$ the field of unit vectors normal to $\partial B(\mathbf{0}, R)$ at $\mathbf{w}(\sigma)$. In this exercise, we will relate the adapted extrinsic surface framing $D = \{\mathbf{N} \times \mathbf{v}, \mathbf{N}, \mathbf{v}\}$ and its Darboux vector $\mathbf{u}$, to the Frenet frame F and its Darboux vector $\kappa \, \mathbf{b} + \tau \, \mathbf{t}$. Note that the components $\mathsf{u}_1 = \mathbf{u} \cdot (\mathbf{N} \times \mathbf{v})$ and $\mathsf{u}_2 = \mathbf{u} \cdot \mathbf{N}$ are respectively called the normal and geodesic curvatures of a curve on the sphere.

1. Show that $\mathbf{N}'(\sigma) = \mathbf{v}(\sigma)/R$.

2. What is then the general form of $\mathbf{u}$ for a curve on a sphere?

3. Show that the curvature κ of $\mathbf{w}$ is given by $\kappa = \sqrt{\mathsf{u}_2^2 + 1/R^2}$.

4. Show that the torsion τ of $\mathbf{w}$ is given by $\tau = \mathsf{u}_2' R / \left(R^2 \, \mathsf{u}_2^2 + 1\right)$. [Hint: use session 3 or session 1.]

5. Compute the geodesic and normal curvatures of a circle of radius r lying on a sphere of radius R.

Note that in general, a curve $\mathbf{w}$ lying on a smooth surface can be equipped with a surface framing; that is one of the director is the normal $\mathbf{N}$ to the surface. In that case the component of the Darboux vector along $\mathbf{N}$ is the *geodesic* curvature of the curve $\mathbf{w}$ and the component along $\mathbf{N} \times \mathbf{v}$ (where $\mathbf{v}$ is the unit tangent to $\mathbf{w}$) is the *normal* curvature of the surface in the direction $\mathbf{v}$.

1.2 Tangent indicatrix

Now consider a general curve $\mathbf{x}(s) \subset \mathbb{R}^3$ parameterised by its arc-length: $\mathbf{x}' = \mathbf{w}(s)$ where $\mathbf{w}$ is a unit vector. This unusual choice of name for the tangent unit to $\mathbf{x}(s)$ allows us to connect with the previous question since $\mathbf{w}(s)$ traces a curve on the unit sphere $\partial B(\mathbf{0}, 1)$. Assume that $||\mathbf{x}''(s)|| \neq 0$ for all s. As in the previous question, σ is the arc-length along the curve $\mathbf{w}$ and $\mathbf{v}$ is the unit tangent to $\mathbf{w}$. Finally, we define K, T and $F^{[x]}$ respectively as the curvature, torsion and Frenet frame of $\mathbf{x}$.

1. Show that $\frac{\mathrm{d}\sigma}{\mathrm{d}s} = K$.

2. What is the matrix of change of basis between $F^{[x]}$ and D?

3. Show that geodesic curvature u_2 (u_2 as defined in 1.1-4) of $\mathbf{w}$ and the torsion T of $\mathbf{x}$ are related by: $\mathsf{u}_2 = T/K$.

2 Generalised Helices

Let $\mathbf{x}(s)$ be a smooth curve parametrised by arclength with non-vanishing curvature. Prove that the tantrix (short for tangent indicatrix) of $\mathbf{x}$ is an arc of a circle if and only if the ratio $\frac{T}{K}$ is constant where K and T are the curvature and torsion of $\mathbf{x}$ respectively.

Such curves are called generalised helices with the case of circular helices, where both curvature and torsion are constant, being a special case. For generalised helices the curvature (and therefore the torsion) need not themselves be constant.

3 Tangent indicatrix of a closed curve

We give two necessary and sufficient conditions so that a closed curve $\mathbf{w}(\sigma)$ lying on the unit sphere is the tantrix of at least one general closed curve $\mathbf{x}$.

1. Show that a closed curve $\mathbf{w}(\sigma) : [a, b] \to \partial B(\mathbf{0}, 1)$ is not contained in a hemisphere if and only if there exists a strictly positive function $f(\sigma)$ such that $\int_a^b f(\sigma)\, \mathbf{w}(\sigma) \mathrm{d}\sigma = \mathbf{0}$.

2. Show that $\mathbf{w}$ (defined as in **1.**) is the tangent indicatrix of a closed curve $\mathbf{x}$ if and only if $\mathbf{w}$ is not contained in a hemisphere.

Is it therefore possible to build two closed curves so that their respective tangent indicatrices do not intersect?

[Hint: You will need Minkowski's hyperplane separation theorem. Let A and B be two disjoint nonempty convex subsets of $\mathbb{R}^n$. Then there exists a nonzero vector $\mathbf{v}$ and a real number c such that

$$\forall \mathbf{x} \in A, \forall \mathbf{y} \in B : \ \mathbf{x} \cdot \mathbf{v} \geq c \ \text{ and } \mathbf{y} \cdot \mathbf{v} \leq c.$$

The hyperplane $H = \{\mathbf{a} \in \mathbb{R}^n : \mathbf{a} \cdot \mathbf{v} = c\}$ separates A and B.]