1 Curves on a sphere

1.1 A general curve lying on a sphere

Consider a curve $w(\sigma)$ lying on the sphere $\partial B(0, R)$ of radius $R$ centred at the origin of $\mathbb{R}^3$. Assume that $\sigma$ is arc-length in $w$: $w'(\sigma) = v(\sigma)$ where $v$ is a unit vector. Also consider $N(\sigma)$ the field of unit vectors normal to $\partial B(0, R)$ at $w(\sigma)$. In this exercise, we will relate the adapted extrinsic surface framing $D = \{N \times v, N, v\}$ and its Darboux vector $u$, to the Frenet frame $F$ and its Darboux vector $\kappa b + \tau t$. Note that the components $u_1 = u \cdot (N \times v)$ and $u_2 = u \cdot N$ are respectively called the normal and geodesic curvatures of a curve on the sphere.

1. Show that $N'(\sigma) = v(\sigma)/R$.

2. What is then the general form of $u$ for a curve on a sphere?

3. Show that the curvature $\kappa$ of $w$ is given by $\kappa = \sqrt{u_2^2 + 1}/R^2$.

4. Show that the torsion $\tau$ of $w$ is given by $\tau = u'_2 R / (R^2 u_2^2 + 1)$. [Hint: use session 3 or session 1.]

5. Compute the geodesic and normal curvatures of a circle of radius $r$ lying on a sphere of radius $R$.

1.2 Tangent indicatrix

Now consider a general curve $x(s) \subset \mathbb{R}^3$ parameterised by its arc-length: $x' = w(s)$ where $w$ is a unit vector. This unusual choice of name for the tangent unit to $x(s)$ allows us to connect with the previous question since $w(s)$ traces a curve on the unit sphere $\partial B(0, 1)$. Assume that $||x''(s)|| \neq 0$ for all $s$. As in the previous question, $\sigma$ is the arc-length along the curve $w$ and $v$ is the unit tangent to $w$. Finally, we define $K$, $T$ and $F^{[x]}$ respectively as the curvature, torsion and Frenet frame of $x$.

1. Show that $\frac{d\sigma}{ds} = K$.

2. What is the matrix of change of basis between $F^{[x]}$ and $D$?

3. Show that geodesic curvature $u_2$ ($u_2$ as defined in 1.1-4) of $w$ and the torsion $T$ of $x$ are related by: $u_2 = T/K$.

2 Generalised Helices

Let $x(s)$ be a smooth curve parametrised by arclength with non-vanishing curvature. Prove that the tantrix (short for tangent indicatrix) of $x$ is an arc of a circle if and only if the ratio $T/K$ is constant where $K$ and $T$ are the curvature and torsion of $x$ respectively.

Such curves are called generalised helices with the case of circular helices, where both curvature and torsion are constant, being a special case. For generalised helices the curvature (and therefore the torsion) need not themselves be constant.
3 Tangent indicatrix of a closed curve

We give two necessary and sufficient conditions so that a closed curve \( w(\sigma) \) lying on the unit sphere is the tantrix of at least one general closed curve \( x \).

1. Show that a closed curve \( w(\sigma) : [a, b] \to \partial B(0, 1) \) is not contained in a hemisphere if and only if there exists a strictly positive function \( f(\sigma) \) such that \( \int_a^b f(\sigma) w(\sigma) \, d\sigma = 0 \).

2. Show that \( w \) (defined as in 1.) is the tangent indicatrix of a closed curve \( x \) if and only if \( w \) is not contained in a hemisphere.

Is it therefore possible to build two closed curves so that their respective tangent indicatrices do not intersect?

[Hint: You will need Minkowski’s hyperplane separation theorem. Let \( A \) and \( B \) be two disjoint nonempty convex subsets of \( \mathbb{R}^n \). Then there exists a nonzero vector \( v \) and a real number \( c \) such that
\[
\forall x \in A, \forall y \in B : x \cdot v \geq c \text{ and } y \cdot v \leq c.
\]
The hyperplane \( H = \{ a \in \mathbb{R}^n : a \cdot v = c \} \) separates \( A \) and \( B \].]