The writhe integral of a single curve $C_1$ (or $x(s)$) is defined as the double integral

$$W_{r}(C_1) = \frac{1}{4\pi} \int_{C_1} \int_{C_1} \frac{(x(\sigma) - x(s)) \cdot (x'(\sigma) \times x'(s))}{||x(\sigma) - x(s)||^3} \, d\sigma \, ds. \tag{1}$$

We denote the integrand of the writhe integral by

$$I_{Wr}(\sigma, s) = \frac{(x(\sigma) - x(s)) \cdot (x'(\sigma) \times x'(s))}{||x(\sigma) - x(s)||^3}.$$

We will discuss the importance and applications of this integral in the next classes. In this exercise set we consider some properties of the Writhe integral.

**Problems**

1. Let $x(t)$ and $y(\xi)$ be two (simply) closed curves such that $x \cap y = \emptyset$. Show that shortest chord between the two curves is doubly critical, that is the shortest chord $d(t_0, \xi_0) = x(t_0) - y(\xi_0)$ is mutually orthogonal to $x'(t_0)$ and $y'(\xi_0)$.

2. Show that the Writhe integral is even about $s = \sigma$, that is $I_{Wr}(\sigma, s) = I_{Wr}(s, \sigma)$ for all $\sigma, s \in C_1$. (Easy, but important.)

3. Show that the Writhe of a curve is invariant under
   - translations $x \mapsto x + a$, $a \in \mathbb{R}^3$,
   - rotations $x \mapsto Rx$, $R \in SO(3)$, and
   - dilations $x \mapsto \lambda x$, $0 < \lambda \in \mathbb{R}$

of the curve. What happens to the Writhe of a curve if $x \mapsto Rx$ with $R \in O(3)$ and $|R| = -1$?

4. Show that the integrand of the Writhe vanishes pointwise for a planar (non-self-intersecting) curve, so that the Writhe of a planar curve is zero. How could you use exercise 3 to prove this last claim without knowing that $I_{Wr}$ vanishes identically in the plane?

5. Suppose that there exists a curve $x(s)$ with $s \in [0, L]$ such that
   - $x'' \in C^1$ and $x''(s) \neq 0$ for all $s$,
   - $x(s) = x(\sigma) \Rightarrow s = \sigma$,
   - The unit vector $e(s, \sigma) = \frac{x(s) - x(\sigma)}{||x(s) - x(\sigma)||}$ satisfies $e \cdot (e_s \times e_\sigma)(s, \sigma) = 0$ for all $s$ and $\sigma$ in $[0, L]$.

Prove that $x$ must then be planar.

6. Define the non-symmetric doublet function $d(s, \sigma)$ as the diameter of the circle through $x(s)$ and $x(\sigma)$ and tangent to the curve at $x(\sigma)$. Show that if $x$ is parameterised by arclength, then

$$W_{r} = \frac{1}{4\pi} \int \int \text{Sign} \left[ e \cdot (e_s \times e_\sigma) \right] \frac{\sin \psi}{d(s, \sigma) d(\sigma, s)} \, ds \, d\sigma, \tag{2}$$

where $\psi$ is the angle between the planes spanned by $e$ and $x'(\sigma)$ and by $e$ and $x'(s)$.
7. Although the course focuses on closed curves, the double integral appearing in the definition (1) of the Writhe of a curve can be computed for open curves as well. Consider the circular helix of radius $r$ and pitch $2\pi p r$ parameterised by

$$\mathbf{x}(s) = (r \cos s, r \sin s, r p s),$$

where $s$ spans $[0, L]$. Note that $s$ is not the arc-length along the curve. By explicitly computing (1), show that the Writhe of $\mathbf{x}$ scales like

$$Wr(\mathbf{x}) \sim \frac{p}{2\pi} \left( \frac{1}{|p|} - \frac{1}{\sqrt{1 + p^2}} \right) L \quad \text{as } L \to +\infty.$$

[Hint: Computing the derivative of $\frac{w}{\sqrt{\sin^2 w + p^2 w^2}}$ w.r.t. $w$ should prove useful.]