**Differential Geometry of Framed Curves**
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Session 7: Exercises A. Flynn

The writhe integral of a single curve $C_1$ (or $\mathbf{x}(s)$) is defined as the double integral

$$Wr(C_1) = \frac{1}{4\pi} \int_{C_1} \int_{C_1} \frac{(\mathbf{x}(\sigma) - \mathbf{x}(s)) \cdot (\mathbf{x}'(\sigma) \times \mathbf{x}'(s))}{||\mathbf{x}(\sigma) - \mathbf{x}(s)||^3} \mathrm{d}\sigma \, \mathrm{d}s. \quad (1)$$

We denote the integrand of the writhe integral by

$$I_{Wr}(\sigma, s) = \frac{(\mathbf{x}(\sigma) - \mathbf{x}(s)) \cdot (\mathbf{x}'(\sigma) \times \mathbf{x}'(s))}{||\mathbf{x}(\sigma) - \mathbf{x}(s)||^3}.$$

We will discuss the importance and applications of this integral in the next classes. In this exercise set we consider some properties of the Writhe integral.

**Problems**

**1.** Let $\mathbf{x}(t)$ and $\mathbf{y}(\xi)$ be two (simply) closed curves such that $\mathbf{x} \cap \mathbf{y} = \emptyset$. Show that shortest chord between the two curves is doubly critical, that is the shortest chord $\mathbf{d}(t_0, \xi_0) = \mathbf{x}(t_0) - \mathbf{y}(\xi_0)$ is mutually orthogonal to $\mathbf{x}'(t_0)$ and $\mathbf{y}'(\xi_0)$.

**2.** Show that the Writhe integral is even about $s = \sigma$, that is $I_{Wr}(\sigma, s) = I_{Wr}(s, \sigma)$ for all $\sigma, s \in C_1$. (Easy, but important.)

**3.** Show that the Writhe of a curve is invariant under

- translations $\mathbf{x} \mapsto \mathbf{x} + \mathbf{a}$, $\mathbf{a} \in \mathbb{R}^3$,
- rotations $\mathbf{x} \mapsto R\mathbf{x}$, $R \in SO(3)$, and
- dilations $\mathbf{x} \mapsto \lambda \mathbf{x}$, $0 < \lambda \in \mathbb{R}$

of the curve. What happens to the Writhe of a curve if $\mathbf{x} \mapsto R\mathbf{x}$ with $R \in O(3)$ and $|R| = -1$?

**4.** Show that the integrand of the Writhe vanishes pointwise for a planar (non-self-intersecting) curve, so that the Writhe of a planar curve is zero. How could you use exercise 2 to prove this last claim without knowing that $I_{\mathrm{Wr}}$ vanishes identically in the plane?

**5.** Suppose that there exists a curve $\mathbf{x}(s)$ with $s \in [0, L]$ such that

- $\mathbf{x}'' \in C^1$ and $\mathbf{x}''(s) \neq 0$ for all $s$,
- $\mathbf{x}(s) = \mathbf{x}(\sigma) \Rightarrow s = \sigma$,
- The unit vector $\mathbf{e}(s, \sigma) = \frac{\mathbf{x}(s) - \mathbf{x}(\sigma)}{||\mathbf{x}(s) - \mathbf{x}(\sigma)||}$ satisfies $\mathbf{e} \cdot \big(\mathbf{e}_s \times \mathbf{e}_\sigma\big)(s, \sigma) = 0$ for all $s$ and $\sigma$ in $[0, L]$.

Prove that $\mathbf{x}$ must then be planar.

**6.** Define the non-symmetric doublet function $d(s, \sigma)$ as the diameter of the circle through $\mathbf{x}(s)$ and $\mathbf{x}(\sigma)$ and tangent to the curve at $\mathbf{x}(\sigma)$. Show that if $\mathbf{x}$ is parameterised by arclength, then

$$Wr = \frac{1}{4\pi} \int\int \mathrm{Sign}\Big[\mathbf{e} \cdot (\mathbf{e}_s \times \mathbf{e}_\sigma)\Big] \, \frac{\sin\psi}{d(s, \sigma)\, d(\sigma,\, s)} \, ds \, d\sigma, \quad (2)$$

where $\psi$ is the angle between the planes spanned by $\mathbf{e}$ and $\mathbf{x}'(\sigma)$ and by $\mathbf{e}$ and $\mathbf{x}'(s)$.

**7.** Although the course focuses on closed curves, the double integral appearing in the definition (1) of the Writhe of a curve can be computed for open curves as well. Consider the circular helix of radius $r$ and pitch $p\,\frac{r}{2\pi}$ parameterised by

$$\mathbf{x}(s) = \big(r\cos s,\; r\sin s,\; r\,p\,s\big),$$

where $s$ spans $[0, L]$. Note that $s$ is not the arc-length along the curve. By explicitly computing (1), show that the Writhe of $\mathbf{x}$ scales like

$$Wr(\mathbf{x}) \sim \frac{p}{2\pi}\left(\frac{1}{|p|} - \frac{1}{\sqrt{1+p^2}}\right) L \qquad \text{as } L \to +\infty.$$

[Hint: Computing the derivative of $\frac{w}{\sqrt{\sin^2 w + p^2 w^2}}$ w.r.t. $w$ should prove useful.]