Proof of the Călugăreanu Theorem

During this week’s lecture, we have seen a proof of Călugăreanu Theorem. Given a $C^3$ closed not self-intersecting curve parameterised by arc-length $x(s) : [0, L] \to \mathbb{R}^3$ and an adapted orthonormal framing $(d_1(s) \ d_2(s) \ d_3(s))$ of $x$, we considered the offset curve $y(s) = x(s) + \epsilon d_1(s)$. The $Lk$ between $x$ and $y$ is then

$$Lk(x, y) = \frac{1}{4\pi} \int_0^L I_{\hat{Lk}}(\sigma) d\sigma, \quad \text{where} \quad I_{\hat{Lk}}(\sigma) = \int_0^L (x(s) - y(\sigma)) \cdot \frac{x'(s) \wedge y'(\sigma)}{||x(s) - y(\sigma)||^3} ds. \quad (1)$$

We also defined the Writhe integrand of $x$ as

$$\hat{I}_{Wr}(\sigma) = \int_0^L (x(s) - x(\sigma)) \cdot \frac{x'(s) \wedge x'(\sigma)}{||x(s) - x(\sigma)||^3} ds. \quad (2)$$

Then we split the domain of integration appearing in $I_{\hat{Lk}}(\sigma)$ into two parts: close to the diagonal, that is on the domain $D_\epsilon = [\sigma - \epsilon^{1/5}, \sigma + \epsilon^{1/5}]$; and away from the diagonal, that is on the domain $D_\epsilon' = [0, L] \setminus D_\epsilon = [0, \sigma - \epsilon^{1/5}] \cup [\sigma + \epsilon^{1/5}, L]$.

We then proved on the one hand that the part away from the diagonal uniformly converges to $\hat{I}_{Wr}$, as $\epsilon \to 0$. On the other hand, we discussed how to prove that the part close to the diagonal uniformly converges to $2u_3(\sigma)$. This exercise session focuses on the nitty-gritty detail of this second part of the proof. Accordingly, in everything that follows, we assume $s \in [\sigma - \epsilon^{1/5}, \sigma + \epsilon^{1/5}]$.

1. We first consider the denominator: show that $||x(s) - y(\sigma)||^3 = \epsilon^3 (1 + \tau^2)^{3/2} (1 + \delta(\tau))^{3/2}$, where $\tau$ is defined by the change of variable $(s - \sigma) = \epsilon \tau$ and where there exists a constant $K_1$ such that $\delta(\tau) < K_1 \epsilon^{1/5}$.

2. Preparing to expand the numerator, compute a Taylor expansion of $x(s)$ about $s = \sigma$ up to second order in $(s - \sigma)$. Also prepare two Taylor expansions of $x'(s)$ respectively to first and second order in $(s - \sigma)$.

3. Show that the numerator in the integrand of $\hat{I}_{\hat{Lk}}(\sigma)$ is equal to the 7 terms listed during the lecture. [Hint: Be careful to use the adequate expansions within the different terms so as to avoid a proliferation of superfluous terms.]

4. Prove that the integral of one of these terms divided by the denominator uniformly converges to $2u_3(\sigma)$ as $\epsilon \to 0$.

5. Prove that the integral of each of the remaining terms divided by the denominator uniformly converges to 0 as $\epsilon \to 0$.

[Hint: You will probably need the following formula

$$\int \frac{\tau^n}{(1 + \tau^2)^{3/2}} d\tau = \frac{\tau}{\sqrt{1 + \tau^2}}, \quad \frac{-1}{\sqrt{1 + \tau^2}} \left\{ \frac{-\tau}{\sqrt{1 + \tau^2}} + \ln \left( \tau + \sqrt{1 + \tau^2} \right) \right\}, \quad \frac{\tau^2 + 2}{\sqrt{1 + \tau^2}},$$

in respectively the cases $n = 0, 1, 2, 3$.]