Proof of the Călugăreanu Theorem

During this week's lecture, we have seen a proof of Călugăreanu Theorem. Given a C^3 closed not self-intersecting curve parameterised by arc-length $\mathbf{x}(s) : [0, L] \mapsto \mathbb{R}^3$ and an adapted orthonormal framing $(\mathbf{d}_1(s)\ \ \mathbf{d}_2(s)\ \ \mathbf{d}_3(s))$ of $\mathbf{x}$, we considered the offset curve $\mathbf{y}(s) = \mathbf{x}(s) + \epsilon\, \mathbf{d}_1(s)$. The Lk between $\mathbf{x}$ and $\mathbf{y}$ is then

$$Lk(\mathbf{x}, \mathbf{y}) = \frac{1}{4\pi} \int_0^L \widehat{I_{Lk}}(\sigma) \mathrm{d}\sigma, \qquad \text{where} \qquad \widehat{I_{Lk}}(\sigma) = \int_0^L \big(\mathbf{x}(s) - \mathbf{y}(\sigma)\big) \cdot \frac{\mathbf{x}'(s) \wedge \mathbf{y}'(\sigma)}{||\mathbf{x}(s) - \mathbf{y}(\sigma)||^3}\, \mathrm{d}s. \tag{1}$$

We also defined the Writhe integrand of $\mathbf{x}$ as

$$\widehat{I_{Wr}}(\sigma) = \int_0^L \big(\mathbf{x}(s) - \mathbf{x}(\sigma)\big) \cdot \frac{\mathbf{x}'(s) \wedge \mathbf{x}'(\sigma)}{||\mathbf{x}(s) - \mathbf{x}(\sigma)||^3}\, \mathrm{d}s. \tag{2}$$

Then we split the domain of integration appearing in $\widehat{I_{Lk}}(\sigma)$ into two parts: close to the diagonal, that is on the domain $D_\epsilon = [\sigma - \epsilon^{1/5}, \sigma + \epsilon^{1/5}]$; and away from the diagonal, that is on the domain $D'_\epsilon = [0, L] \setminus D_\epsilon = [0, \sigma - \epsilon^{1/5}] \cup [\sigma + \epsilon^{1/5}, L]$.

We then proved on the one hand that the part away from the diagonal uniformly converges to $\widehat{I_{Wr}}$ as $\epsilon \to 0$. On the other hand, we discussed how to prove that the part close to the diagonal uniformly converges to $2\,\mathsf{u}_3(\sigma)$. This exercise session focuses on the nitty-gritty detail of this second part of the proof. Accordingly, in everything that follows, we assume $s \in [\sigma - \epsilon^{1/5}, \sigma + \epsilon^{1/5}]$.

1. We first consider the denominator: show that $||\mathbf{x}(s) - \mathbf{y}(\sigma)||^3 = \epsilon^3\, (1 + \tau^2)^{3/2}\, (1 + \delta(\tau))^{3/2}$, where τ is defined by the change of variable $(s - \sigma) = \epsilon\, \tau$ and where there exists a constant K_1 such that $\delta(\tau) < K_1 \epsilon^{1/5}$.

2. Preparing to expand the numerator, compute a Taylor expansion of $\mathbf{x}(s)$ about $s = \sigma$ up to second order in $(s - \sigma)$. Also prepare two Taylor expansions of $\mathbf{x}'(s)$ respectively to first and second order in $(s - \sigma)$.

3. Show that the numerator in the integrand of $\widehat{I_{Lk}}(\sigma)$ is equal to the 7 terms listed during the lecture. [Hint: Be careful to use the adequate expansions within the different terms so as to avoid a proliferation of superfluous terms.]

4. Prove that the integral of one of these terms divided by the denominator uniformly converges to $2\,\mathsf{u}_3(\sigma)$ as $\epsilon \to 0$.

5. Prove that the integral of each of the remaining terms divided by the denominator uniformly converges to 0 as $\epsilon \to 0$.

[Hint: You will probably need the following formula

$$\int \frac{\tau^n}{(1 + \tau^2)^{\frac{3}{2}}}\, d\tau = \frac{\tau}{\sqrt{1 + \tau^2}}, \frac{-1}{\sqrt{1 + \tau^2}}, \left\{ \frac{-\tau}{\sqrt{1 + \tau^2}} + \ln\left(\tau + \sqrt{1 + \tau^2}\right) \right\}, \frac{\tau^2 + 2}{\sqrt{1 + \tau^2}}.$$

in respectively the cases $n = 0, 1, 2, 3$.]