1 Framings of closed curves (revisiting material from last lecture)

Let \( x : s \in [0, L] \mapsto x(s) \in \mathbb{R}^3 \) be a regular, closed curve. Assume throughout that for any closed framing of \( x \) the CFW theorem holds. We have only proven this for \( C^3 \) curves although results with weaker assumptions are known.

1. Show that there exists a closed (adapted) framing of \( x \).

   a) Pick a framing, any framing. That is in fact, following question 3 in session 2, pick a function \( u_3(s) \) and an initial condition for the third component of the Darboux vector.
   
   b) Show how you can pick a different function \( U_3 \) so that the associated framing is closed.

2. We defined a Writhe framing as a framing the twist of which is \( u_3(s) = -\frac{1}{2} \int_0^L \text{wr}(s, \sigma) \, d\sigma \). Show that a Writhe framing is always closed.

   a) Define the register angle\(^1\) \( \varphi(s) \) between a Writhe framing and the closed framing of question 1.
   
   b) Use \( Lk = Tw + Wr \) to show that the difference of register between the beginning and end of \( x \) must be an integer multiple of \( 2\pi \).

3. Show that a Writhe framing always has zero \( Lk \).

4. a) Show that any (adapted) framing of a curve is closed if and only if \( Tw + Wr \) is an integer.
   
   b) Show that the natural framings of a closed curve are closed if and only if the writhe is an integer.
   
   c) More generally, show that if \( D(s) \) is a natural framing of \( x \), then \( D(L) = D(0) R_{3, \varphi} \) where

\[
R_{3, \varphi} = \begin{pmatrix}
\cos \varphi & -\sin \varphi & 0 \\
\sin \varphi & \cos \varphi & 0 \\
0 & 0 & 1
\end{pmatrix}, \quad \text{and} \quad \varphi = 2\pi \left( \text{Wr}(x) \mod 1 \right).
\]

This suggests a method to accurately calculate writhe without having to compute a double integral.

Turn the page ...

\(^1\)This is the angle between two framings.
2 Writhe of a figure 8 curve (this is an example more than an exercise)

Consider the figure of eight curve
\[
x(t) = \begin{pmatrix}
\cos t \\
\sin 2t \\
\frac{2}{\nu} \sin t
\end{pmatrix},
\]
represented in Fig. 1. Note that when \( \nu = 0 \), \( x \) self-intersects so that Writhe is not defined. Here we study the limiting cases of \( \nu \to 0 \) from either side.

By direct application of the double integral definition of Writhe, you could show (but beware: this is somewhat tedious) that
\[
Wr(x; \nu) = \frac{\nu}{\pi} \int_0^\pi \frac{\cos \mu \, d\mu}{\left(\nu^2 \cos^2 \frac{\mu}{2} + \sin^2 \frac{\mu}{2}\right) \sqrt{\nu^2 \cos^2 \frac{\mu}{2} + \sin^2 \frac{\mu}{2} + \cos^2 \mu}},
\]
(2)

1. Show that for all \( \nu \in (0, 1) \) and for all \( \mu \in (0, \pi/2) \), we have
\[
\frac{\cos \mu}{\sqrt{\nu^2 \cos^2 \frac{\mu}{2} + \sin^2 \frac{\mu}{2} + \cos^2 \mu}} \leq 1,
\]
(3)
\[ 1 - \left( \nu^2 \cos^2 \frac{\mu}{2} + \sin^2 \frac{\mu}{2} \right) \leq \frac{1}{\sqrt{\nu^2 \cos^2 \frac{\mu}{2} + \sin^2 \frac{\mu}{2} + \cos^2 \mu}}. \quad (4) \]

2. Accordingly, show that
\[ \lim_{\nu \to 0^+} W_r(x; \nu) = 1. \quad (5) \]

3. What can you say about \( \lim_{\nu \to 0^-} W_r(x; \nu) \)?

It is also possible to (numerically) compute the Writhe framing of each of these curves and those are displayed on Fig. 1.

3 Writhe Framing of a trefoil

![Figure 2: Writhe framing of a trefoil knot.](image)

A Writhe framing of a trefoil is shown in Fig. 2. Show that \( Lk = 0 \) by counting crossings.

4 Writhe of an offset curve

Given a regular, closed curve \( x : s \in [a, b] \mapsto x(s) \in \mathbb{R}^3 \) and \( X(t) = (d_1(s) \ d_2(s) \ d_3(s)) \) an adapted and closed framing of \( x(s) \), let \( u(s) = u_i(s) \ d_i(s) \) be the Darboux vector of \( X(s) \) and define the offset curve
\[ z(s) = x(s) + \eta \ d_1(s), \quad (6) \]

where \( \eta \) is sufficiently small such that for all \( \epsilon \in (0, \eta] \) there is no intersection between the curves \( x(s) \) and the curve \( x(s) + \epsilon \ d_1(s) \). Note that whilst \( z \) is an offset curve of \( x \), \( x \) is also an offset curve of \( z \).

The aim of this exercise is to prove that
\[ W_r(z) = W_r(x) + \frac{1}{2\pi} \int_a^b u_3(s) \left( 1 - \frac{||x'(s)||}{||z'(s)||} \right) ds. \quad (7) \]

This can be done in essentially two steps:

1. Recall from question 3, session 3 that you know a particular adapted framing of \( z \) and compute the third component of the Darboux vector of that frame.

2. Apply \( Lk = Tw + W_r \) twice together with the fact that \( Lk(x, z) = Lk(z, x) \).