1 Euler-Rodrigues parameters

As discussed during last lecture, there are many ways to parameterize a general proper rotation matrix \( Q \in SO(3) \) using only three or four parameters. Here we further study the four-dimensional parametrization in terms of Euler-Rodrigues parameters. Any element \( q \in \mathbb{R}^4 \) satisfying the condition

\[
q \cdot q = q_0^2 + q_1^2 + q_2^2 + q_3^2 = 1
\]

(1)

can be interpreted as a right-handed rotation of angle \( \phi \) and the axis of which is along the unit vector \( \mathbf{n} \), where \( \phi \) and \( \mathbf{n} \) solve

\[
\cos \phi/2 = q_0, \quad \text{and} \quad \mathbf{n} \sin \phi/2 = \begin{pmatrix} q_1 \\ q_2 \\ q_3 \end{pmatrix}.
\]

Note that the condition (1) also means that \( q \in S^3 \), where \( S^3 \) is the unit sphere in \( \mathbb{R}^4 \).

(a) Show that the rotation matrix \( Q \in SO(3) \) associated with \( q \) is

\[
Q(q) = \text{Id} + 2(q_0 \mathbf{q}^\times + \mathbf{q}^\times \mathbf{q}^\times) = (q_0^2 - \mathbf{q} \cdot \mathbf{q}) \text{Id} + 2 \mathbf{q} \otimes \mathbf{q} + 2q_0 \mathbf{q}^\times,
\]

(2)

where \( \mathbf{q} = (q_1 q_2 q_3)^T \). Expand the very RHS of (2) to show that

\[
Q(q) = \begin{pmatrix}
q_0^2 - q_1^2 - q_2^2 + q_3^2 & 2(q_1 q_2 - q_3 q_0) & 2(q_1 q_3 + q_2 q_0) \\
2(q_1 q_2 + q_3 q_0) & -q_1^2 + q_2^2 - q_3^2 + q_0^2 & 2(q_0 q_3 - q_2 q_1) \\
2(q_1 q_3 - q_2 q_0) & 2(q_1 q_0 + q_2 q_3) & -q_1^2 - q_2^2 + q_3^2 + q_0^2
\end{pmatrix}.
\]

(3)

The formula (3) is a direct parameterisation of \( SO(3) \) by elements of \( S^3 \). Next, we check that the matrix \( Q(q) \) indeed has two expected properties.

(b) Given \( q \in S^3 \), \( Q(q) \in SO(3) \) defined as in (a) and \( \mathbf{q} = (q_1 q_2 q_3)^T \), assume that \( \mathbf{q} \neq 0 \) and show that \( Q(q) \mathbf{q} = \mathbf{q} \); that is \( \mathbf{q} \) defines the axis of rotation of \( Q \). What can be said about \( Q \) if \( \mathbf{q} = 0? \)

(c) Let \( \theta \) be the argument of one of the complex eigenvalue of \( Q(q) \). Show that \( \cos(\theta/2) = \pm q_0 \).

Finally, here are a couple of results for later use

(d) Show that the application (3) is 2-to-1. That is any element \( q \in S^3 \) corresponds to a single element element of \( SO(3) \) but for any element of \( R \in SO(3) \) there are exactly two elements of \( S^3 \) sent on \( R \) by (3). What is the relation between those two elements of \( S^3 \).

(e) Given \( q \in S^3 \), show that \( \{q, B_1 q, B_2 q, B_3 q\} \) is an orthonormal basis for \( \mathbb{R}^4 \), where the matrices \( B_i \) \( (i = 1, 2, 3) \) are defined by

\[
B_1 = \begin{pmatrix}
0 & -1 & 0 & 0 \\
1 & 0 & 0 & 0 \\
0 & 0 & 0 & 1 \\
0 & 0 & -1 & 0
\end{pmatrix},
\]

(4)

\[
B_2 = \begin{pmatrix}
0 & 0 & -1 & 0 \\
0 & 0 & 0 & -1 \\
1 & 0 & 0 & 0 \\
0 & 1 & 0 & 0
\end{pmatrix},
\]

(5)
and
\[
B_3 = \begin{pmatrix}
0 & 0 & 0 & -1 \\
0 & 0 & 1 & 0 \\
0 & -1 & 0 & 0 \\
1 & 0 & 0 & 0
\end{pmatrix}.
\]
(6)

(f) Show that (3) can be factorised as \( Q(q) = F(q)B(q)^T \), where \( F \) and \( B \) are appropriate \( 3 \times 4 \) matrices each of whose entries is a linear function of the components of \( q(s) \).

2 Composition rule for Euler-Rodrigues parameters

(a) Given two Euler parameters \( p = \begin{pmatrix} p_0 \\ p \end{pmatrix}, q = \begin{pmatrix} q_0 \\ q \end{pmatrix} \), we define \( B(q) := (q, B(q)^T) \in \mathbb{R}^{4 \times 4} \), where \( B(q) \in \mathbb{R}^{3 \times 4} \) was derived in exercise session 12. Show that
\[
B(q)p = p_0q + \sum_{i=1}^{3} p_i B_i q,
\]
where the matrices \( B_i, i = 1, 2, 3 \) have been given in session 12 and read as
\[
B_1 = \begin{pmatrix}
0 & -1 & 0 & 0 \\
1 & 0 & 0 & 0 \\
0 & 0 & 0 & 1 \\
0 & 0 & -1 & 0
\end{pmatrix}, \quad
B_2 = \begin{pmatrix}
0 & 0 & -1 & 0 \\
0 & 0 & 0 & -1 \\
1 & 0 & 0 & 0 \\
0 & 1 & 0 & 0
\end{pmatrix}, \quad
B_3 = \begin{pmatrix}
0 & 0 & 0 & -1 \\
0 & 0 & 1 & 0 \\
0 & -1 & 0 & 0 \\
1 & 0 & 0 & 0
\end{pmatrix}.
\]

(b) Given two Euler parameters \( p = \begin{pmatrix} p_0 \\ p \end{pmatrix}, q = \begin{pmatrix} q_0 \\ q \end{pmatrix} \), we define \( F(q) := (q, F(q)^T) \in \mathbb{R}^{4 \times 4} \), where \( F(q) \in \mathbb{R}^{3 \times 4} \) was derived in exercise session 12. Show that
\[
F(q)p = p_0q + \sum_{i=1}^{3} p_i F_i q,
\]
where the matrices \( F_i, i = 1, 2, 3 \) are here defined as
\[
F_1 = \begin{pmatrix}
0 & -1 & 0 & 0 \\
1 & 0 & 0 & 0 \\
0 & 0 & 0 & -1 \\
0 & 0 & 1 & 0
\end{pmatrix}, \quad
F_2 = \begin{pmatrix}
0 & 0 & -1 & 0 \\
0 & 0 & 0 & 1 \\
1 & 0 & 0 & 0 \\
0 & -1 & 0 & 0
\end{pmatrix}, \quad
F_3 = \begin{pmatrix}
0 & 0 & 0 & -1 \\
0 & 0 & -1 & 0 \\
0 & 1 & 0 & 0 \\
1 & 0 & 0 & 0
\end{pmatrix}.
\]

(c) If \( Q(q) \) is the rotation matrix parametrized by \( q \) and \( v = \begin{pmatrix} 0 \\ v \end{pmatrix} \), show that
\[
F^T(q)B(q)v = \begin{pmatrix}
0 \\
Q(q)v
\end{pmatrix}.
\]

(d) Using the previous points, show the composition rule for Euler parameters. Namely, if \( Q(q) = Q(p)Q(p^*) \), then \( q \) can be written in terms of \( p \) and \( p^* \) as
\[
q = p_0^*p + \sum_{i=1}^{3} p_i^* B_i p.
\]
3 Composition rule for Cayley vectors

Show that if \( \text{Cay}(c^x) = \text{Cay}(u^x)\text{Cay}(u^{*x}) \), then \( c \) can be written in terms of \( u \) and \( u^* \) as

\[
c = \frac{u + u^* + u \times u^*}{1 - u \cdot u^*}.
\]

Moreover, prove that the denominator vanishes if and only if \( \text{Cay}(u^x)\text{Cay}(u^{*x}) \) corresponds to a rotation by \( \pi \). Hint: use the stereographic projection in order to exploit the composition rule for Euler parameters.

4 Darboux vector in terms of Euler-Rodrigues parameters

Consider a curve \( Q(s) \in SO(3), s \in (0, L) \), parametrized by the curve of Euler parameters \( q(s) \in \mathbb{R}^4, s \in (0, L) \). Show that

\[
u(s) = 2B(q(s))q'(s),
\]

where \( u(s) \in \mathbb{R}^3 \) is the Darboux vector in the directors (moving) frame satisfying \( Q'(s) = Q(s)u(s)^x \).

5 Euler-Rodrigues parameters of particular framed curves in \( SO(3) \)

5.1 Euler-Rodrigues parameters associated with the Frenet frame of a helix

Come back to the helix defined during the first question of the exercise session 1:

\[
\alpha(s) = \left( a \cos \frac{s}{c}, a \sin \frac{s}{c}, b \right), \tag{7}
\]

where \( a > 0, b > 0 \) and \( c = \sqrt{a^2 + b^2} \).

The Frenet frame \( F(s) \) of \( \alpha(s) \) is a curve in \( SO(3) \). Show that it can be described by the Euler parameters \( q(s) \) such that

\[
q_0(s) = -\sin \frac{s}{2c} \sqrt{\frac{c + b}{2c}}
\]

and

\[
q(s) = \left( -\sin \frac{s}{2c} \sqrt{\frac{c - b}{2c}}, \cos \frac{s}{2c} \sqrt{\frac{c - b}{2c}}, \cos \frac{s}{2c} \sqrt{\frac{c + b}{2c}} \right)^T. \tag{8}
\]

Hint: it is possible to show the following composition rule for Euler parameters (we will derive it in the next exercise session). Let us assume that \( Q(q) = Q(p)Q(p^*) \), then \( q \) can be written in terms of \( p \) and \( p^* \) as

\[
q = p^*p + \sum_{i=1}^{3} p_i^* B_i p. \tag{9}
\]

5.2 Euler-Rodrigues parameters of the multiply covered circle

1. What are the Euler-Rodrigues parameters describing a twist-less adapted frame to a \( n \) times covered circle (i.e the Frenet-Serret frame of a helix with 0 pitch)?

2. What are the Euler-Rodrigues parameters describing an adapted frame to a \( n \) times covered circle if the local twist of the frame is prescribed by a function \( u_3(s) \) where \( s \) is the arc-length along the circle.