Preliminaries

In this exercise session we focus on the special case of inextensible tubes with a uniform, circular, and orthogonal cross-section. We will call such a tube an *ideal* tube for short. An ideal tube $\mathcal{T}$ is therefore an object embedded in $\mathbb{R}^3$ defined as

$$\mathcal{T} = \{\mathbf{z} = \mathbf{x}(s) + y_1\,\mathbf{d}_1 + y_2\,\mathbf{d}_2 \in \mathbb{R}^3 : s \in [0,\, L],\ (y_1, y_2) \in \mathcal{B}(0,\, R)\}, \tag{1}$$

where $\mathbf{x}$ is an $\mathbb{R}^3$ curve equipped with an adapted frame $\{\mathbf{d}_i\}$ such that $\mathbf{x}' = \mathbf{d}_3$, L is a positive constant that we call the length of the tube, and $\mathcal{B}(0,\, R)$ is the disk centred at the origin of $\mathbb{R}^2$ and of radius R.

Such a tube is said to be uniform because the domain of $(y_1,\, y_2)$ is independent of s, circular because said domain is a disk, and orthogonal because the vector from the centreline $\mathbf{x}(s)$ to any point $\mathbf{z}(s,\, y_1,\, y_2)$ in the corresponding section is perpendicular to the tangent $\mathbf{d}_3(s)$ to $\mathbf{x}(s)$.

A tube such as $\mathcal{T}$ can be deformed into different configurations. Because we assumed that it is inextensible, it can however not be stretched: in particular, a *material* parameterisation – that is a labelling of its material points – of its centreline by arc-length in any given configuration will remain an arc-length parameterisation in all other configurations. We also note that forces can be applied to it and that under such loadings, at equilibrium the sum of all forces on any of its part vanishes.

In ideal tubular *strings* (or ideal strings for short), there exists a positive scalar function $T(s) > 0$ such that the material points corresponding to $s \in (s^\star,\, L)$ apply a force $T(s^\star)\,\mathbf{d}_3(s^\star)$ on the material points corresponding to $s \in [0,\, s^\star)$. We have seen in the lecture that at equilibrium, such a tube satisfies force balance iff

$$(T\,\mathbf{x}')' + \mathbf{f} = \mathbf{0}, \tag{2}$$

where $'$ denotes derivation w.r.t. s and $\mathbf{f}$ is a density of external force per unit s.

We have also seen during the lecture that when the external loading is due to a frictionless contact force from a surface, as for instance the boundary of another tube,

$$\mathbf{f} \cdot \mathbf{x}' = \mathbf{0}. \tag{3}$$

1 Ideal string under frictionless contact

Assuming (3), show that

1. $T(s)$ is constant along the string, and

2. $\mathbf{f} = -T\,\kappa\,\mathbf{n}$ where κ is the curvature of $\mathbf{x}$.

2 Two strings with a single contact line

Now consider two strings with respective centrelines $\mathbf{x}_1$ and $\mathbf{x}_2$ parameterised by their respective arc-lengths s and σ and in contact along a single line. Show that the force balance on both strings implies that

$$T_1\,\mathbf{x}_1' + T_2\,\dot{\mathbf{x}}_2 = \boldsymbol{\xi}, \tag{4}$$

where T_i are the tensions in each strings, $'$ and $\dot{()}$ denote derivatives w.r.t s and σ, and $\boldsymbol{\xi}$ is a constant vector representing the total force acting across the two cross-sections.

3 There exist equilibria of pairs of ideal strings with circular helical centrelines

Show that given two ideal strings the sections of which have respective radii R_1 and R_2, there exist equilibrium configurations of the pair such that each centreline forms a circular helix. Compute these equilibria and in particular, find T_1 and T_2.

4 The centrelines of a pair of ideal strings in equilibria form generalised helices

Show that (4) implies that $\mathbf{x}_1' \cdot \boldsymbol{\xi}$ and $\dot{\mathbf{x}}_2 \cdot \boldsymbol{\xi}$ are constant. Hence, exercise 7 from session 1 guarantees that κ_i/τ_i is constant for $i \in \{1, 2\}$.

5 The centrelines of a pair of ideal strings in equilibrium are Bertrand mates

Show that the centrelines of a pair of ideal strings in equilibrium with a single contact line have a common (up to a sign) principal normal. Such curves are called Bertrand mates.

6 Bertrand mates

This exercise is reproduced from [1]. Let $\mathbf{x}(t)$ be a regular and parameterised (not necessarily by arclength) curve with non-vanishing curvature $\kappa(t)$ and torsion $\tau(t)$. It is called a *Bertrand curve* iff there exists a *Bertrand mate* $\overline{\mathbf{x}}(t)$, that is a curve such that the normal line to $\mathbf{x}$ at t is also the normal line to $\overline{\mathbf{x}}$ at t. We can write

$$\overline{\mathbf{x}}(t) = \mathbf{x}(t) + r\,\mathbf{n}(t), \tag{5}$$

where $\mathbf{n}$ is the principal normal to $\mathbf{x}$. In the language of this course, $\overline{\mathbf{x}}$ is a particular offset curve of $\mathbf{x}$. Show that

1. r is independent of t.

2. $\mathbf{x}$ is a Bertrand curve iff there exists two constants A and B such that

$$A\,\kappa(t) + B\,\tau(t) = 1. \tag{6}$$

3. If $\mathbf{x}$ has more than one Bertrand mate, it has infinitely many Bertrand mate. This case occurs iff $\mathbf{x}$ is a circular helix.

7 All equilibria are circular helices

Using questions 4, 5, and 6, show that all equilibria of pairs of ideal strings with a single contact line are pairs of circular helices.

8 Points of closest approach for double helices (Optional)

Let $\mathbf{x}$ and $\mathbf{y}$ be a pair of diametrically opposed circular helices with equal radii R. Show that, for each point on $\mathbf{y}$, there is a single point of closest approach on $\mathbf{x}$, namely its diametrically opposed point, if and only if the angle θ between $\mathbf{x}'(0)$ and $\mathbf{y}'(0)$ is less than $\frac{\pi}{2}$.

References

[1] M. P. Do Carmo. *Differential Geometry of Curves and Surfaces.* Prentiss-Hall Inc., 1976.