1 Properties of skew symmetric matrices

1. Let $\mathbf{u}, \mathbf{v} \in \mathbb{R}^3$. The vector product $\mathbf{u} \times \mathbf{v}$, in components, reads:

$$\mathbf{u} \times \mathbf{v} = \begin{bmatrix} u_2v_3 - u_3v_2 \\ u_3v_1 - u_1v_3 \\ u_1v_2 - u_2v_1 \end{bmatrix} \tag{1}$$

From the equality above one can see that the following skew symmetric matrix

$$[\mathbf{u}\times] = \begin{bmatrix} 0 & -u_3 & u_2 \\ u_3 & 0 & -u_1 \\ -u_2 & u_1 & 0 \end{bmatrix} \tag{2}$$

satisfies $\mathbf{u} \times \mathbf{v} = [\mathbf{u}\times]\mathbf{v}$. The mapping $\mathbf{u} \leftrightarrow [\mathbf{u}\times]$ is by inspection linear and invertible.

2. Given any two vectors $\boldsymbol{v}$ and $\boldsymbol{w}$, we can compute the following:

$$\begin{aligned}
\boldsymbol{v}^T[M\boldsymbol{u}\times]\boldsymbol{w} &= \boldsymbol{v} \cdot (M\boldsymbol{u} \times \boldsymbol{w}) \\
&= \left|\left[\boldsymbol{v} \vdots M\boldsymbol{u} \vdots \boldsymbol{w}\right]\right| \\
&= |M| \left|\left[M^{-1}\boldsymbol{v} \vdots \boldsymbol{u} \vdots M^{-1}\boldsymbol{w}\right]\right| \\
&= |M|(M^{-1}\boldsymbol{v}) \cdot (\boldsymbol{u} \times M^{-1}\boldsymbol{w}) \\
&= |M|(M^{-1}\boldsymbol{v})^T[\boldsymbol{u}\times]M^{-1}\boldsymbol{w} \\
&= |M|\boldsymbol{v}^T\left(M^{-T}[\boldsymbol{u}\times]M^{-1}\right)\boldsymbol{w} \\
&= \boldsymbol{v}^T\left(|M|M^{-T}[\boldsymbol{u}\times]M^{-1}\right)\boldsymbol{w}
\end{aligned}$$

As this is true for any $\boldsymbol{v}$ and $\boldsymbol{w}$, we can conclude that $[M\mathbf{u}\times] = |M|M^{-T}[\boldsymbol{u}\times]M^{-1}$. In these computations we denoted by $\left[\mathbf{a} \vdots \mathbf{b} \vdots \mathbf{c}\right]$ the three by three matrix whose columns are the vectors $\boldsymbol{a}$, $\boldsymbol{b}$, and $\boldsymbol{c}$.

When $M \in SO(3)$, we have $[M\mathbf{u}\times] = M[\boldsymbol{u}\times]M^T$.

3. For any $\mathbf{v} \in \mathbb{R}^3$, we have

$$[\mathbf{u}\times]^2\mathbf{v} = [\mathbf{u}\times]\left([\mathbf{u}\times]\,\mathbf{v}\right) = \mathbf{u} \times (\mathbf{u} \times \mathbf{v}) = (\mathbf{u}\cdot\mathbf{v})\mathbf{u} - (\mathbf{u}\cdot\mathbf{u})\mathbf{v} = \left(\mathbf{u}\otimes\mathbf{u} - ||\mathbf{u}||^2\mathrm{I}\right)\mathbf{v}. \tag{3}$$

Since (25) holds for all $\mathbf{v}$ we have that $[\mathbf{u}\times]^2 = \mathbf{u}\otimes\mathbf{u} - ||\mathbf{u}||^2\mathrm{I}$. Or just carry out the matrix multiplication $[\mathbf{u}\times][\mathbf{u}\times]$ and identify terms.

4. For a given matrix

$$A = \begin{bmatrix} -\lambda & -u_3 & u_2 \\ u_3 & -\lambda & -u_1 \\ -u_2 & u_1 & -\lambda \end{bmatrix} \tag{4}$$

$$\begin{aligned} |A| &= -\lambda(\lambda^2+u_1^2)+u_3(-\lambda u_3-u_1u_2)+u_2(u_3u_1-\lambda u_2) & (5)\\ &= -\lambda^3-|\mathbf{u}|^2\lambda & (6)\end{aligned}$$

So here

$$P(\lambda) = -\lambda^3-|\mathbf{u}|^2\lambda \tag{7}$$

which has roots $\lambda = 0, \pm i|\mathbf{u}|$. Then verify that $[\mathbf{u}\times]^3+|\mathbf{u}|^2[\mathbf{u}\times]{=}0\in\mathbb{R}^{3\times3}$

2 Rotations in three dimensions

1. From the properties of the scalar product, $\forall\mathbf{w}\in\mathbb{R}^3$

$$\|Q\mathbf{w}\|^2 = Q\mathbf{w}\cdot Q\mathbf{w} = \mathbf{w}\cdot Q^TQ\mathbf{w} = \|\mathbf{w}\|^2 \tag{8}$$

where we have used the fact that Q is a rotation matrix, i.e. $Q^TQ = I$. If now λ is an eigenvalue for Q, let $\mathbf{w}$ be the corresponding eigenvector

$$\|Q\mathbf{w}\| = \|\lambda\mathbf{w}\| = |\lambda|\|\mathbf{w}\| \tag{9}$$

but then from equation (8) we obtain

$$|\lambda| = 1 \tag{10}$$

and therefore λ lies on the unit circle in the complex plane.

2. Note that the complex conjugate $\overline{\lambda}$ is an eigenvalue of Q (with corresponding eigenvector $\overline{\mathbf{w}}$) whenever λ is an eigenvalue of Q (with corresponding eigenvector $\mathbf{w}$). This follows from $\overline{Q} = Q$. Explicitly,

$$Q\mathbf{w} = \lambda\mathbf{w} \Rightarrow Q\overline{\mathbf{w}} = \overline{Q\mathbf{w}} = \overline{\lambda\mathbf{w}} = \overline{\lambda}\overline{\mathbf{w}}.$$

As Q has exactly 3 eigenvalues (counted according to multiplicity), we obtain from (10) that the set of eigenvalues of Q is

$$\mathrm{eig}(Q) = \{e^{ix}, e^{-ix}, 1\} \quad \text{or} \quad \mathrm{eig}(Q) = \{e^{ix}, e^{-ix}, -1\}$$

for some $x\in[0,\pi]$. But for $\mathrm{eig}(Q) = \{e^{ix}, e^{-ix}, -1\}$ we get[1] $\det(Q) = -1$, so that we must have

$$\mathrm{eig}(Q) = \{e^{ix}, e^{-ix}, 1\} \quad \text{for } x\in[0,\pi]. \tag{11}$$

This also shows that $\lambda = 1$ cannot have multiplicity 2, but only 1 or 3. If $\lambda = 1$ has multiplicity 3, then $Q = \mathrm{Id}$, which therefore is the only case where there can be a non-unique axis of rotation.

For any eigenvalue λ of Q, the inverse λ^{-1} will be an eigenvalue of $Q^{-1} = Q^T$ corresponding to the same eigenvector. Therefore, $Q\mathbf{w} = \mathbf{w}$ immediately implies both that $\mathbf{w}$ is parallel (or

[1] Here, we make use of

$$\det(A) = \prod_{i=1}^{n}\lambda_i,$$

for any matrix $A\in\mathbb{C}^{n\times n}$ with eigenvalues $\lambda_1, \lambda_2, ..., \lambda_n$ repeated according to (algebraic) multiplicity.

anti parallel) to the axis of rotation and $\mathbf{w}$ is an element of the nullspace of $S = (Q - Q^T)$, i.e. $S\mathbf{w} = 0$. On the other hand if $\mathbf{u}$ is the unique axial vector of the skew matrix S, then

$$0 = S\mathbf{w} = \mathbf{u} \times \mathbf{w}, \tag{12}$$

so that $\mathbf{u}$ and $\mathbf{w}$ are parallel (or anti parallel), i.e. the axial vector of the skew matrix $(Q-Q^T)$ is parallel (or anti parallel) to $\mathbf{w}$.

3. If $\mathbf{v}$ is any vector orthogonal to the axis of rotation $\mathbf{w}$, then

$$Q\mathbf{v} \cdot \mathbf{w} = \mathbf{v} \cdot Q^T\mathbf{w} = \mathbf{v} \cdot \mathbf{w} = 0$$

Furthermore from (8) we can conclude that $\|\mathbf{v}\| = \|Q\mathbf{v}\|$.

On the one hand, the angle θ between $Q\mathbf{v}$ and $\mathbf{v}$ obeys

$$Q\mathbf{v} \cdot \mathbf{v} = \|Q\mathbf{v}\| \, \|\mathbf{v}\| \cos\theta \overset{(8)}{=} \|\mathbf{v}\|^2 \cos\theta. \tag{13}$$

On the other hand the trace of a matrix is the sum of its eigenvalues[2]. Then (11) implies

$$\operatorname{tr}(Q) = 1 + 2\cos x. \tag{14}$$

In the special cases $x = 0$ and $x = \pi$, the proof is immediate since $\cos x = \pm 1$ and $\mathbf{v}$ is itself an eigenvector and $Q\mathbf{v} = \pm\,\mathbf{v}$ so that $\cos\theta = \pm 1$.

Otherwise, $x \in (0, \pi)$ and we will show that $\cos x = \cos\theta$ holds in general. Along the way, we will need a number of properties regarding the complex eigenvectors of Q. We list them hereunder with their proof. Once and for all, $x \in (0, \pi)$ and $\mathbf{z} \in \mathbb{C}^3$ is a norm 1 eigenvector of Q corresponding to the eigenvalue $e^{i\,x}$ from question 1.2. The hermitian product between complex vectors $\mathbf{a}$, $\mathbf{b} \in \mathbb{C}^3$ is noted $\langle \mathbf{a}, \mathbf{b} \rangle := \bar{\mathbf{a}}\mathbf{b}$. The scalar product between real vectors $\mathbf{x}$, $\mathbf{y} \in \mathbb{R}^3$ is noted $\mathbf{x} \cdot \mathbf{y}$.

Proposition 1. *The conjugate $\overline{\mathbf{z}}$ of $\mathbf{z}$ is an eigenvector of Q with eigenvalue $e^{-i\,x}$.*

Proof. See exercise 2.2. □

Proposition 2. *If $x \in (0, \pi)$, the eigenvector $\mathbf{z}$ is such that $\langle \mathbf{z}, \overline{\mathbf{z}} \rangle = 0$.*

Proof. $e^{-i\,x} \langle \mathbf{z}, \overline{\mathbf{z}} \rangle = \langle e^{i\,x}\, \mathbf{z}, \overline{\mathbf{z}} \rangle = \langle Q\, \mathbf{z}, \overline{\mathbf{z}} \rangle = \langle \mathbf{z}, Q^T\, \overline{\mathbf{z}} \rangle = \langle \mathbf{z}, e^{i\,x}\, \overline{\mathbf{z}} \rangle = e^{i\,x} \langle \mathbf{z}, \overline{\mathbf{z}} \rangle$ and whenever $x \in (0, \pi)$, this implies $\langle \mathbf{z}, \overline{\mathbf{z}} \rangle = 0$. □

Proposition 3. *Let $\mathbf{x}$, $\mathbf{y} \in \mathbb{R}^3$ be respectively the real and imaginary part of the eigenvector $\mathbf{z} = \mathbf{x} + i\,\mathbf{y}$. If $x \in (0, \pi)$, then $\mathbf{x}$ and $\mathbf{y}$ are orthogonal: $\mathbf{x} \cdot \mathbf{y} = 0$. Furthermore,*

$$\mathbf{x} \cdot \mathbf{x} = \mathbf{y} \cdot \mathbf{y} = 1/2. \tag{15}$$

Proof. On the one hand, from Proposition 2 we have

$$\begin{aligned} 0 = \langle \mathbf{z}, \overline{\mathbf{z}} \rangle = \langle \mathbf{x} + i\,\mathbf{y}, \mathbf{x} - i\,\mathbf{y} \rangle &= \Big(\mathbf{x} \cdot \mathbf{x} - \mathbf{y} \cdot \mathbf{y}\Big) - 2\,i\,\Big(\mathbf{x} \cdot \mathbf{y}\Big), \\ \Rightarrow \mathbf{x} \cdot \mathbf{x} = \mathbf{y} \cdot \mathbf{y}, &\qquad \text{and} \qquad \mathbf{x} \cdot \mathbf{y} = 0. \end{aligned} \tag{16}$$

On the other hand, $\|\mathbf{z}\| = 1$ so that $\langle \mathbf{z}, \mathbf{z} \rangle = \mathbf{x} \cdot \mathbf{x} + \mathbf{y} \cdot \mathbf{y} = 1 \overset{(16)}{\Rightarrow} \mathbf{x} \cdot \mathbf{x} = \mathbf{y} \cdot \mathbf{y} = \frac{1}{2}$. □

[2] This comes from the fact that if $A \in \mathbb{R}^{n\times n}$ there exists $P \in SU(n)$ such that $P^{-1}AP$ is diagonal. Then $\operatorname{tr}(PAP^{-1})$ is the sum of the eigenvalues of A. But the trace is invariant under cyclic perturbations since $\operatorname{tr}(ABC) = A_{ij}B_{jk}C_{ki} = B_{jk}C_{ki}A_{ij} = \operatorname{tr}(BCA)$. So $\operatorname{tr}(P^{-1}AP) = \operatorname{tr}(APP^{-1}) = \operatorname{tr} A$.

Now we are ready to proceed with the exercise. We have already seen the the real eigenvector is providing the axis of rotation, but we still haven't say anything about the role of the two complex conjugate eigenvectors. We are going to show that the real and imaginary part of these vectors are providing a base for the plane orthogonal to the axis of rotation. Reminding (13), it is possible to give explicitly a relation between the angle of rotation θ and the real number x appearing in the eigenvalues just taking $\mathbf{v} = \mathbf{x}$ or $\mathbf{v} = \mathbf{y}$. For example let's consider the case $\mathbf{v} = \mathbf{x}$.

Since Q is real, we have

$$e^{ix}\langle \mathbf{w}, \mathbf{z}\rangle = \langle Q\mathbf{w}, Q\mathbf{z}\rangle = \langle \mathbf{w}, Q^T Q\mathbf{z}\rangle = \langle \mathbf{w}, \mathbf{z}\rangle \Rightarrow \langle \mathbf{w}, \mathbf{z}\rangle = 0 \text{ and}$$

$$\langle \mathbf{w}, \mathbf{z}\rangle = \langle \mathbf{w}, \mathbf{x} + i\mathbf{y}\rangle = \mathbf{w}\cdot\mathbf{x} + i\mathbf{w}\cdot\mathbf{y} = 0 \Rightarrow \mathbf{w}\cdot\mathbf{x} = 0 \text{ and } \mathbf{w}\cdot\mathbf{y} = 0.$$

For (13) with $\mathbf{v} = \mathbf{x}$ and Proposition 3 it follows that

$$Q\mathbf{x}\cdot\mathbf{x} = \|\mathbf{x}\|^2\cos(\theta) = \frac{\cos(\theta)}{2}.$$

Since Q is linear

- $Q\mathbf{x} + iQ\mathbf{y} = Q(\mathbf{x} + i\mathbf{y}) = Q\mathbf{z} = e^{ix}\mathbf{z} = e^{ix}(\mathbf{x} + i\mathbf{y})$,
- $Q\mathbf{x} - iQ\mathbf{y} = Q(\mathbf{x} - i\mathbf{y}) = Q\bar{\mathbf{z}} = e^{-ix}\mathbf{z} = e^{-ix}(\mathbf{x} - i\mathbf{y})$.

Using Proposition 3 again we finally find

- $Q\mathbf{x}\cdot\mathbf{x} + iQ\mathbf{y}\cdot\mathbf{x} = e^{ix}\|\mathbf{x}\|^2 = \dfrac{e^{ix}}{2}$,
- $Q\mathbf{x}\cdot\mathbf{x} - iQ\mathbf{y}\cdot\mathbf{x} = e^{-ix}\|\mathbf{x}\|^2 = \dfrac{e^{-ix}}{2}$.

Summing up the previous two expression we have $Q\mathbf{x}\cdot\mathbf{x} = \frac{\cos(x)}{2}$ and $\cos(x) = \cos(\theta)$.

4. If $Q \in SO(3)$ is a rotation about a unit vector $\mathbf{n} \in \mathbb{R}^3$ in counter clock wise sense (from right handed rule) of angle $\theta \in (0,\ \pi]$ (if $\theta = 0$ there is non-unique axis of rotation), then $Q\mathbf{u} = \mathbf{u}\cos(\theta) + (\mathbf{n}\times\mathbf{u})\sin(\theta)$ if $\mathbf{u} \perp \mathbf{n}$, $\|\mathbf{u}\| = 1$ ($\{\mathbf{n}, \mathbf{u}, \mathbf{n}\times\mathbf{u}\}$ is a right-handed orthonormal basis of $\mathbb{R}^3$)

$$\Longrightarrow\ (Q - Q^T)\mathbf{u} = \cos(\theta)\mathrm{I}\mathbf{u} - \cos(\theta)\mathrm{I}\mathbf{u} + \sin(\theta)[\mathbf{n}\times]\mathbf{u} + \sin(\theta)[\mathbf{n}\times]\mathbf{u} = 2\sin(\theta)[\mathbf{n}\times]\mathbf{u},$$

and, referring to Question 1.1, $[Q - Q^T]_{12} = -2\sin(\theta)n_3$ with $\sin(\theta) > 0$ if $\theta \in (0,\ \pi)$, and the unit vector $\mathbf{n}$ is automatically recovered from the axial vector of $(Q - Q^T)$.

5. The idea of the proof is to decompose any vector $\mathbf{v} \in \mathbb{R}^3$, different from $\mathbf{n}$, in a component along $\mathbf{n}$ and an other component orthogonal to $\mathbf{n}$. We then apply the rotation Q to the vector $\mathbf{v}$. We have that $\mathbf{v}$ can be written in the following manner:

$$\mathbf{v} = (\mathbf{n}\cdot\mathbf{v})\mathbf{n} + (\mathbf{n}\times\mathbf{v})\times\mathbf{n} \tag{17}$$

We then apply to (17) the rotation Q:

$$Q\mathbf{v} = (\mathbf{n}\cdot\mathbf{v})\mathbf{n} + \sin(\phi)(\mathbf{n}\times\mathbf{v}) + \cos(\phi)(\mathbf{n}\times\mathbf{v})\times\mathbf{n} \tag{18}$$

Next, we recall two properties of a double cross product

$$\begin{aligned}(\mathbf{n}\times\mathbf{v})\times\mathbf{n} &= -\mathbf{n}\times(\mathbf{n}\times\mathbf{v}) && (19)\\ \mathbf{n}\times(\mathbf{n}\times\mathbf{v}) &= [\mathbf{n}\times]^2\mathbf{v} = (\mathbf{n}\otimes\mathbf{n}-||\mathbf{n}||^2\mathbb{I})\mathbf{v} = (\mathbf{n}\cdot\mathbf{v})\mathbf{n}-||\mathbf{n}||^2\mathbf{v} && (20)\end{aligned}$$

Using (19) and (20) in (18) we obtain

$$\begin{aligned}Q\mathbf{v} &= ||\mathbf{n}||^2\mathbf{v}+\sin(\phi)(\mathbf{n}\times\mathbf{v})+(1-\cos(\phi))[\mathbf{n}\times]^2\mathbf{v} && (21)\\ &= \mathbf{v}+\sin(\phi)[\mathbf{n}\times]\mathbf{v}+(1-\cos(\phi))[\mathbf{n}\times]^2\mathbf{v} && (22)\end{aligned}$$

As the latter expression is valid for any vector $\mathbf{v}$ we obtain that the matrix Q can be expressed as

$$Q = \mathrm{I}+\sin(\phi)[\mathbf{n}\times]+(1-\cos(\phi))[\mathbf{n}\times]^2. \tag{23}$$

Finally, using $[\mathbf{n}\times]^2 = \mathbf{n}\otimes\mathbf{n}-||\mathbf{n}||^2\mathbb{I}$ we get

$$Q = \cos(\phi)\mathrm{I}+\sin(\phi)[\mathbf{n}\times]+(1-\cos(\phi))\mathbf{n}\otimes\mathbf{n} \tag{24}$$

Now for any $\mathbf{v}\in\mathbb{R}^3$, we have

$$(\mathbf{n}^\times)^2\mathbf{v} = \mathbf{n}^\times\,\mathbf{n}^\times\,\mathbf{v} = \mathbf{n}^\times\,(\mathbf{n}^\times\,\mathbf{v}) = \mathbf{n}\times(\mathbf{n}\times\mathbf{v}) = (\mathbf{n}\cdot\mathbf{v})\mathbf{n}-\mathbf{v} = (\mathbf{n}\otimes\mathbf{n}-\mathrm{Id})\,\mathbf{v}. \tag{25}$$

Since (25) holds for all $\mathbf{v}$ we have $\mathbf{n}\otimes\mathbf{n} = \mathrm{Id}+(\mathbf{n}^\times)^2$ which can be substituted in Eq. (2) from the question sheet.

3 Computing matrix exponential using Cayley-Hamilton

Taylor series for exponential of matrix M is

$$exp(M) = I+M+\frac{M^2}{2!}+\frac{M^3}{3!}+\frac{M^4}{4!}+\frac{M^5}{5!}+\frac{M^6}{6!}+\frac{M^7}{7!}+\frac{M^8}{8!}+\cdots \tag{26}$$

and in the case $M = [\mathbf{u}\times]$ we have characteristic polynomial (from (7)) as

$$\begin{aligned}[\mathbf{u}\times]^3+|\mathbf{u}|^2[\mathbf{u}\times] &= 0\\ \implies [\mathbf{u}\times]^3 &= -|\mathbf{u}|^2[\mathbf{u}\times]\end{aligned}$$

After substituting above equation in (26) and collecting terms of $[\mathbf{u}\times]$ and $[\mathbf{u}\times]^2$ implies

$$\begin{aligned}exp\left([\mathbf{u}\times]\right) &= I+\left(1-\frac{|\mathbf{u}|^2}{3!}+\frac{|\mathbf{u}|^4}{5!}-\frac{|\mathbf{u}|^6}{7!}+\cdots\right)[\mathbf{u}\times]+\left(\frac{1}{2!}-\frac{|\mathbf{u}|^2}{4!}+\frac{|\mathbf{u}|^4}{6!}-\frac{|\mathbf{u}|^6}{8!}+\cdots\right)[\mathbf{u}\times]^2\\ &= I+\frac{|\mathbf{u}|-\frac{|\mathbf{u}|^3}{3!}+\frac{|\mathbf{u}|^5}{5!}-\frac{|\mathbf{u}|^7}{7!}+\cdots}{|\mathbf{u}|}[\mathbf{u}\times]+\frac{\frac{|\mathbf{u}|^2}{2!}-\frac{|\mathbf{u}|^4}{4!}+\frac{|\mathbf{u}|^6}{6!}-\frac{|\mathbf{u}|^8}{8!}+\cdots}{|\mathbf{u}|^2}[\mathbf{u}\times]^2\\ &= I+\frac{\sin(|\mathbf{u}|)}{|\mathbf{u}|}[\mathbf{u}\times]+\frac{1-\cos(|\mathbf{u}|)}{|\mathbf{u}|^2}[\mathbf{u}\times]^2\\ &= I+\sin(|\mathbf{u}|)\frac{[\mathbf{u}\times]}{|\mathbf{u}|}+[1-\cos(|\mathbf{u}|)]\left[\frac{[\mathbf{u}\times]}{|\mathbf{u}|}\right]^2. && (27)\end{aligned}$$

With $\phi = |\mathbf{u}|$ and $\mathbf{n} = \frac{\mathbf{u}}{|\mathbf{u}|}$, finally we have

$$exp\left([\mathbf{u}\times]\right) = I+\sin(\phi)[\mathbf{n}\times]+(1-\cos(\phi))[\mathbf{n}\times]^2 \tag{28}$$

which is (5) in the annonce.