1 Properties of skew symmetric matrices

1. Let \( u, v \in \mathbb{R}^3 \). The vector product \( u \times v \), in components, reads:

\[
\begin{bmatrix}
-u_3 v_2 - u_2 v_3 \\
u_3 v_1 - u_1 v_3 \\
u_1 v_2 - u_2 v_1
\end{bmatrix}
\]

From the equality above one can see that the following skew symmetric matrix

\[
[u \times] =
\begin{bmatrix}
0 & u_2 & -u_3 \\
-u_2 & 0 & u_1 \\
u_3 & -u_1 & 0
\end{bmatrix}
\]

satisfies \( u \times v = [u \times]v \). The mapping \( u \mapsto [u \times] \) is by inspection linear and invertible.

2. Given any two vectors \( v \) and \( w \), we can compute the following:

\[
v^T [Mu \times] w = v \cdot (Mu \times w) = 
\begin{bmatrix}
v : Mu : w
\end{bmatrix} = 
\begin{bmatrix}
|M| \left[ M^{-1}v : u : M^{-1}w \right]
\end{bmatrix} = 
\begin{bmatrix}
|M| (M^{-1}v) \cdot (u \times M^{-1}w) = 
\begin{bmatrix}
|v^T (M^{-T}[u \times]M^{-1}) w = 
\begin{bmatrix}
|v^T (|M|M^{-T}[u \times]M^{-1}) w
\end{bmatrix}
\end{bmatrix}
\end{bmatrix}
\]

As this is true for any \( v \) and \( w \), we can conclude that \( [Mu \times] = |M|M^{-T}[u \times]M^{-1} \). In these computations we denoted by \( [a : b : c] \) the three by three matrix whose columns are the vectors \( a, b, \) and \( c \).

When \( M \in SO(3) \), we have \( [Mu \times] = M[u \times]M^T \).

3. For any \( v \in \mathbb{R}^3 \), we have

\[
[u \times]^2 v = [u \times]([u \times]v) = u \times (u \times v) = (u \cdot v)u - (u \cdot u)v = (u \otimes u - ||u||^2 I) v.
\]

Since (3) holds for all \( v \) we have that \( [u \times]^2 = u \otimes u - ||u||^2 I \). Or just carry out the matrix multiplication \( [u \times][u \times] \) and identify terms.

4. For a given matrix

\[
A =
\begin{bmatrix}
-\lambda & -u_3 & u_2 \\
u_3 & -\lambda & -u_1 \\
-u_2 & u_1 & -\lambda
\end{bmatrix}
\]
\[ |A| = -\lambda(\lambda^2 + u_1^2) + u_3(-\lambda u_3 - u_1 u_2) + u_2(u_3 u_1 - \lambda u_2) \quad (5) \]
\[ = -\lambda^3 - |u|^2 \lambda \quad (6) \]

So here

\[ P(\lambda) = -\lambda^3 - |u|^2 \lambda \quad (7) \]

which has roots \( \lambda = 0, \pm i|u| \). Then verify that \([u \times]^3 + |u|^2[u \times] = 0 \in \mathbb{R}^{3 \times 3}\)

2 Rotations in three dimensions

1. From the properties of the scalar product, \( \forall \mathbf{w} \in \mathbb{R}^3 \) and any \( Q \in SO(3) \)

\[ \|Q\mathbf{w}\|^2 = Q\mathbf{w} \cdot Q\mathbf{w} = \mathbf{w} \cdot Q^T Q \mathbf{w} = \|\mathbf{w}\|^2 \quad (8) \]

where we have used the fact that \( Q \) is a rotation matrix, i.e. \( Q^T Q = I \). If now \( \lambda \) is an eigenvalue for \( Q \) and \( \mathbf{w} \) the corresponding eigenvector

\[ \|Q\mathbf{w}\| = \|\lambda \mathbf{w}\| = \lambda \|\mathbf{w}\| \quad (9) \]

but then from equation (8) we obtain

\[ |\lambda| = 1 \quad (10) \]

and therefore \( \lambda \) lies on the unit circle in the complex plane.

2. Note that the complex conjugate \( \overline{\lambda} \) is an eigenvalue of \( Q \) (with corresponding eigenvector \( \overline{\mathbf{w}} \)) whenever \( \lambda \) is an eigenvalue of \( Q \) (with corresponding eigenvector \( \mathbf{w} \)). This follows from \( \overline{Q} = Q \). Explicitly,

\[ Q\mathbf{w} = \lambda \mathbf{w} \Rightarrow \overline{Q}\overline{\mathbf{w}} = \overline{\lambda \mathbf{w}} = \overline{\lambda} \overline{\mathbf{w}}. \]

As \( Q \) has exactly 3 eigenvalues (counted according to multiplicity), we obtain from (10) that the set of eigenvalues of \( Q \) is

\[ \text{eig}(Q) = \{e^{ix}, e^{-ix}, 1\} \quad \text{or} \quad \text{eig}(Q) = \{e^{ix}, e^{-ix}, -1\} \]

for some \( x \in [0, \pi] \). But for \( \text{eig}(Q) = \{e^{ix}, e^{-ix}, -1\} \) we get\(^1\) \( \det(Q) = -1 \), so that we must have

\[ \text{eig}(Q) = \{e^{ix}, e^{-ix}, 1\} \quad \text{for} \quad x \in [0, \pi]. \quad (11) \]

This also shows that \( \lambda = 1 \) cannot have multiplicity 2, but only 1 or 3. If \( \lambda = 1 \) has multiplicity 3, then \( Q = \text{Id} \), which therefore is the only case where there can be a non-unique axis of rotation.

For any eigenvalue \( \lambda \) of \( Q \), the inverse \( \lambda^{-1} \) will be an eigenvalue of \( Q^{-1} = Q^T \) corresponding to the same eigenvector. Therefore, \( Q\mathbf{w} = \mathbf{w} \) immediately implies that \( \mathbf{w} \) is an element of the

\(^1\)Here, we make use of

\[ \det(A) = \prod_{i=1}^{n} \lambda_i, \]

for any matrix \( A \in \mathbb{C}^{n \times n} \) with eigenvalues \( \lambda_1, \lambda_2, ..., \lambda_n \) repeated according to (algebraic) multiplicity.
nullspace of \( S = (Q - Q^T) \), i.e. \( Sw = 0 \). On the other hand if \( z \) is the unique axial vector of the skew matrix \( S \) (i.e. \( S = [z \times] \)), then

\[
0 = Sw = z \times w,
\]

so that \( z \) and \( w \) are parallel, i.e. the axial vector of the skew matrix is parallel to the axis of rotation of \( Q \).

3. If \( v \) is any vector orthogonal to the axis of rotation \( w \), then

\[
Qv \cdot w = v \cdot Q^Tw = v \cdot w = 0.
\]

Furthermore from (8) we can conclude that if \( v \) is a unit vector, then so is \( Qv \).

On the one hand, the angle \( \theta \) between \( Qv \) and \( v \) obeys

\[
Qv \cdot v = \|Qv\| \|v\| \cos \theta \overset{(8)}{=} \cos \theta.
\]

On the other hand the trace of a matrix is the sum of its eigenvalues\(^2\). Then (11) implies

\[
\text{tr}(Q) = 1 + 2 \cos x.
\]

In the special cases \( x = 0 \) and \( x = \pi \), the proof is immediate since \( \cos x = \pm 1 \) and \( v \) is itself an eigenvector and \( Qv = \pm v \) so that \( \cos \theta = \pm 1 \).

Otherwise, \( x \in (0, \pi) \) and we will show that \( \cos x = \cos \theta \) holds in general. Along the way, we will need a number of properties regarding the complex eigenvectors of \( Q \). We list them hereunder with their proof. Once and for all, \( x \in (0, \pi) \) and \( z \in \mathbb{C}^3 \) is a norm 1 eigenvector of \( Q \) corresponding to the eigenvalue \( e^{ix} \) from question 1.2. The hermitian product between complex vectors \( a, b \in \mathbb{C}^3 \) is noted \( \langle a, b \rangle \). The scalar product between real vectors \( x, y \in \mathbb{R}^3 \) is noted \( x \cdot y \).

**Proposition 1.** The conjugate \( \overline{z} \) of \( z \) is an eigenvector of \( Q \) with eigenvalue \( e^{-ix} \).

**Proof.** See exercise 2.2.

**Proposition 2.** If \( x \in (0, \pi) \), the eigenvector \( z \) is such that \( \langle z, \overline{z} \rangle = 0 \).

**Proof.** Compute

\[
e^{-ix} \langle z, \overline{z} \rangle = \langle e^{ix}z, \overline{z} \rangle = \langle Qz, \overline{z} \rangle = \langle z, Q^T \overline{z} \rangle = \langle z, e^{ix} \overline{z} \rangle = e^{ix} \langle z, \overline{z} \rangle.\]

And whenever \( x \in (0, \pi) \), Eq. (16) implies \( \langle z, \overline{z} \rangle = 0 \).

**Proposition 3.** Let \( x, y \in \mathbb{R}^3 \) be respectively the real and imaginary part of the unit eigenvector \( z = x + iy \). If \( x \in (0, \pi) \), then \( x \) and \( y \) are orthogonal: \( x \cdot y = 0 \). Furthermore,

\[
x \cdot x = y \cdot y = 1/2.
\]

\(^2\)This comes from the fact that if \( A \in \mathbb{R}^{n \times n} \) there exists \( P \in SU(n) \) such that \( P^{-1}AP \) is diagonal. Then \( \text{tr}(PAP^{-1}) \) is the sum of the eigenvalues of \( A \). But the trace is invariant under cyclic perturbations since \( \text{tr}(ABC) = A_{ij}B_{jk}C_{ki} = B_{jk}C_{ki}A_{ij} = \text{tr}(BCA) \). So \( \text{tr}(P^{-1}AP) = \text{tr}(APP^{-1}) = \text{tr}A \).
Proof. On the one hand, from Proposition 2 we have

\[ 0 = \langle z, z \rangle = \langle x + i y, x - i y \rangle = (x \cdot x - y \cdot y) - 2i(x \cdot y), \]
\[ \Rightarrow x \cdot x = y \cdot y, \quad \text{and} \quad x \cdot y = 0. \tag{18} \]

On the other hand, \( z \) is of norm 1 so that

\[ \langle z, z \rangle = x \cdot x + y \cdot y = 1 \quad (\Rightarrow x \cdot x = y \cdot y = \frac{1}{2}). \]

Now we are ready to proceed with the exercise. First we show that there exists a complex number \( a \in \mathbb{C} \) such that \( v = a z + \overline{a} \mathbf{z} \). Indeed, since \( z \) and \( \mathbf{z} \) span the space orthogonal to \( w \) in \( \mathbb{C}^3 \), there exist complex numbers \( a, b \in \mathbb{C} \) such that \( v = a z + b z \). But then with \( x \) and \( y \) defined as in Proposition 3, the fact that \( v \) is a real vector implies

\[ (\text{Im}(a) + \text{Im}(b))x + (\text{Re}(a) - \text{Re}(b))y = 0 \Rightarrow b = \overline{a}, \tag{19} \]

where the implication is due to Proposition 3: \( x \) and \( y \) are orthogonal and of non zero norm so that both brackets must vanish independently.

Then the fact that \( v \) is a unit vector implies that \( |a|^2 = 1/2 \). Indeed, \( 1 = \langle v, v \rangle = \langle az + \overline{a} \mathbf{z}, az + \overline{a} \mathbf{z} \rangle = |a|^2 + |\overline{a}|^2 = 2|a|^2 \), where the third equality is due to Proposition 2.

Finally, simply compute

\[ \cos \theta \overset{(14)}{=} Qv \cdot v = \langle Q(a z + \overline{a} \mathbf{z}), az + \overline{a} \mathbf{z} \rangle = \langle a e^{ix} z + \overline{a} e^{-ix} \mathbf{z}, az + \overline{a} \mathbf{z} \rangle, \]
\[ \overset{\text{Prop.}(3)}{=} |a|^2(e^{-ix} + e^{ix}) = \frac{e^{-ix} + e^{ix}}{2} = \cos x. \tag{20} \]

4. Eigenvectors can always be multiplied by a non zero scalar and they remain eigenvectors. In particular the scalar can be complex. In the case of \( Q \in SO(3) \), this means that any real vector perpendicular to the axial vector \( u \) can be taken as the real part \( x \) of the eigenvector with complex eigenvalue \( e^{i\theta} \). The eigenvalue relation in terms of real and imaginary parts of the eigenvector \( z = x + iy \) becomes

\[ Qz = e^{i\theta}(x + iy) = (\cos(\theta)x - \sin(\theta)y) + i(\sin(\theta)x + \cos(\theta)y). \tag{21} \]

Then since \( Qz = Q(x + iy) \)

\[ Qx = \cos(\theta)x - \sin(\theta)y, \]
\[ Qy = \sin(\theta)x + \cos(\theta)y. \tag{22} \]

From proposition 3, we know that \( |y| = |x| \) & \( y \cdot x = 0 \). Thus, for real and imaginary parts of the eigenvectors we can take any \( x \) with \( x \cdot u = 0 \) and

\[ y = \pm(u \times x). \tag{23} \]

According to the right-hand rule if \( Q \) is a counter-clockwise rotation through the angle \( \phi \) about \( u \), and \( a = u \times v \)

\[ Qx = \cos(\phi)x + \sin(\phi)a, \]
\[ Qa = -\sin(\phi)x + \cos(\phi)a. \tag{24} \]
By comparing (22) with the relations according to the right-hand rule (eqn. 24), we find
\( \phi = -\theta \) and \( y = + (u \times x) \) the choice of imaginary part of eigenvector for which \( \phi > 0 \) corresponds to an anti-clockwise direction about \( u \) and \( \{u, x, y\} \) is right-handed.

Now fix an axial vector \( u \) and consider an arbitrary vector \( v \in \mathbb{R}^3 \) which can be written as

\[
v = (u \cdot v)u + (I - u \otimes u)v. \tag{25}
\]

Then we can consider \( x = (I - u \otimes u)v \) as the real part of the complex eigenvector and \( y = u \times ((I - u \otimes u)v) \) as the imaginary part. Then with \( \phi \) being a counter-clockwise relation about \( u \) we find

\[
Qv = (u \cdot v)Qu + Q(I - u \otimes u)v \tag{26}
\]

\[
= (u \cdot v)u + \cos(\phi)(I - u \otimes u)v + \sin(\phi)(u \times (I - u \otimes u)v) \tag{27}
\]

\[
= (1 - \cos(\phi))(u \otimes u)v + \sin(\phi)(u \times v) + \cos(\phi)v \tag{28}
\]

which gives (3) in the announce, because \( v \) is arbitrary. Then (2) follows using 1.3.