1 Properties of skew symmetric matrices

1. Let $u, v \in \mathbb{R}^3$. The vector product $u \times v$, in components, reads:

$$u \times v = \begin{bmatrix} u_2v_3 - u_3v_2 \\ u_3v_1 - u_1v_3 \\ u_1v_2 - u_2v_1 \end{bmatrix}$$ (1)

From the equality above one can see that the following skew symmetric matrix

$$[u \times] = \begin{bmatrix} 0 & -u_3 & u_2 \\ u_3 & 0 & -u_1 \\ -u_2 & u_1 & 0 \end{bmatrix}$$ (2)

satisfies $u \times v = [u \times]v$. The mapping $u \mapsto [u \times]$ is by inspection linear and invertible.

2. Given any two vectors $v$ and $w$, we can compute the following:

$$v^T [Mu \times]w = v \cdot (Mu \times w)$$

$$= \left| v : Mu : w \right|$$

$$= |M| \left| (M^{-1}v) : u : M^{-1}w \right|$$

$$= |M| (M^{-1}v) \cdot (u \times M^{-1}w)$$

$$= |M| (v^T (M^{-T}[u \times]M^{-1}) w)$$

$$= v^T (|M|M^{-T}[u \times]M^{-1}) w$$

As this is true for any $v$ and $w$, we can conclude that $[Mu \times] = |M|M^{-T}[u \times]M^{-1}$. In these computations we denoted by $\begin{bmatrix} a : b : c \end{bmatrix}$ the three by three matrix whose columns are the vectors $a, b,$ and $c$.

When $M \in SO(3)$, we have $[Mu \times] = M[u \times]M^T$.

3. For any $v \in \mathbb{R}^3$, we have

$$[u \times]^2v = [u \times]([u \times]v) = u \times (u \times v) = (u \cdot v)u - (u \cdot u)v = (u \otimes u - \|u\|^2I)v.$$ (3)

Since (3) holds for all $v$ we have that $[u \times]^2 = u \otimes u - \|u\|^2I$. Or just carry out the matrix multiplication $[u \times][u \times]$ and identify terms.

4. For a given matrix

$$A = \begin{bmatrix} -\lambda & -u_3 & u_2 \\ u_3 & -\lambda & -u_1 \\ -u_2 & u_1 & -\lambda \end{bmatrix}$$ (4)
\[ |A| = -\lambda (\lambda^2 + u_1^2) + u_3(-\lambda u_3 - u_1u_2) + u_2(u_3u_1 - \lambda u_2) \]
\[ = -\lambda^3 - |u|^2 \lambda \] (5)

So here
\[ P(\lambda) = -\lambda^3 - |u|^2 \lambda \] (6)

which has roots \( \lambda = 0, \pm i|u| \). Then verify that \([u \times]^3 + |u|^2[u \times] = 0 \in \mathbb{R}^{3 \times 3}\)

2 Rotations in three dimensions

1. From the properties of the scalar product, \( \forall w \in \mathbb{R}^3 \) and any \( Q \in SO(3) \)
\[ \|Qw\|^2 = Qw \cdot Qw = w \cdot Q^T Qw = \|w\|^2 \] (8)
where we have used the fact that \( Q \) is a rotation matrix, i.e. \( Q^T Q = I \). If now \( \lambda \) is an eigenvalue for \( Q \) and \( w \) the corresponding eigenvector
\[ \|Qw\| = \|\lambda w\| = |\lambda| \|w\| \] (9)
but then from equation (8) we obtain
\[ |\lambda| = 1 \] (10)
and therefore \( \lambda \) lies on the unit circle in the complex plane.

2. Note that the complex conjugate \( \bar{\lambda} \) is an eigenvalue of \( Q \) (with corresponding eigenvector \( \bar{w} \)) whenever \( \lambda \) is an eigenvalue of \( Q \) (with corresponding eigenvector \( w \)). This follows from \( \bar{Q} = Q \). Explicitly,
\[ Qw = \lambda w \Rightarrow \bar{Q}w = \bar{\lambda}w = \bar{\lambda} \bar{w}. \]
As \( Q \) has exactly 3 eigenvalues (counted according to multiplicity), we obtain from (10) that the set of eigenvalues of \( Q \) is
\[ \text{eig}(Q) = \{e^{ix}, e^{-ix}, 1\} \quad \text{or} \quad \text{eig}(Q) = \{e^{ix}, e^{-ix}, -1\} \]
for some \( x \in [0, \pi] \). But for \( \text{eig}(Q) = \{e^{ix}, e^{-ix}, -1\} \) we get \( \text{det}(Q) = -1 \), so that we must have
\[ \text{eig}(Q) = \{e^{ix}, e^{-ix}, 1\} \quad \text{for} \quad x \in [0, \pi]. \] (11)
This also shows that \( \lambda = 1 \) cannot have multiplicity 2, but only 1 or 3. If \( \lambda = 1 \) has multiplicity 3, then \( Q = \text{Id} \), which therefore is the only case where there can be a non-unique axis of rotation.

For any eigenvalue \( \lambda \) of \( Q \), the inverse \( \lambda^{-1} \) will be an eigenvalue of \( Q^{-1} = Q^T \) corresponding to the same eigenvector. Therefore, \( Qw = w \) immediately implies that \( w \) is an element of the
nullspace of $S = (Q - Q^T)$, i.e. $Sw = 0$. On the other hand if $z$ is the unique axial vector of the skew matrix $S$ (i.e. $S = [z \times]$), then

$$0 = Sw = z \times w,$$

so that $z$ and $w$ are parallel, i.e. the axial vector of the skew matrix is parallel to the axis of rotation of $Q$.

3. If $v$ is any vector orthogonal to the axis of rotation $w$, then

$$Qv \cdot w = v \cdot Q^Tw = v \cdot w = 0.$$  

Furthermore from (8) we can conclude that if $v$ is a unit vector, then so is $Qv$.

On the one hand, the angle $\theta$ between $Qv$ and $v$ obeys

$$Qv \cdot v = \|Qv\| \|v\| \cos \theta \overset{(8)}{=} \cos \theta.$$  

On the other hand the trace of a matrix is the sum of its eigenvalues\(^2\). Then (11) implies

$$\text{tr}(Q) = 1 + 2 \cos x.$$  

In the special cases $x = 0$ and $x = \pi$, the proof is immediate since $\cos x = \pm 1$ and $v$ is itself an eigenvector and $Qv = \pm v$ so that $\cos \theta = \pm 1$.

Otherwise, $x \in (0, \pi)$ and we will show that $\cos x = \cos \theta$ holds in general. Along the way, we will need a number of properties regarding the complex eigenvectors of $Q$. We list them hereunder with their proof. Once and for all, $x \in (0, \pi)$ and $z \in \mathbb{C}^3$ is a norm 1 eigenvector of $Q$ corresponding to the eigenvalue $e^{ix}$ from question 1.2. The hermitian product between complex vectors $a, b \in \mathbb{C}^3$ is noted $\langle a, b \rangle$. The scalar product between real vectors $x, y \in \mathbb{R}^3$ is noted $x \cdot y$.

**Proposition 1.** The conjugate $\bar{z}$ of $z$ is an eigenvector of $Q$ with eigenvalue $e^{-ix}$.

**Proof.** See exercise 2.2. \qed

**Proposition 2.** If $x \in (0, \pi)$, the eigenvector $z$ is such that $\langle z, \bar{z} \rangle = 0$.

**Proof.** Compute

$$e^{-ix} \langle z, \bar{z} \rangle = \langle e^{ix}z, \bar{z} \rangle = \langle Qz, \bar{z} \rangle = \langle z, Q^T\bar{z} \rangle = \langle z, e^{ix}\bar{z} \rangle = e^{ix} \langle z, \bar{z} \rangle.$$  

And whenever $x \in (0, \pi)$, Eq. (16) implies $\langle z, \bar{z} \rangle = 0$. \qed

**Proposition 3.** Let $x, y \in \mathbb{R}^3$ be respectively the real and imaginary part of the unit eigenvector $z = x + iy$. If $x \in (0, \pi)$, then $x$ and $y$ are orthogonal: $x \cdot y = 0$. Furthermore,

$$x \cdot x = y \cdot y = 1/2.$$  

\(^2\)This comes from the fact that if $A \in \mathbb{R}^{n \times n}$ there exists $P \in SU(n)$ such that $P^{-1}AP$ is diagonal. Then $\text{tr}(PAP^{-1})$ is the sum of the eigenvalues of $A$. But the trace is invariant under cyclic perturbations since $\text{tr}(ABC) = A_{ij}B_{jk}C_{ki} = B_{jk}C_{ki}A_{ij} = \text{tr}(BCA)$. So $\text{tr}(P^{-1}AP) = \text{tr}(APP^{-1}) \overset{\text{tr}}{=} \text{tr}A$.  

3
Proof. On the one hand, from Proposition 2 we have
\[ 0 = (z, \bar{z}) = (x + iy, x - iy) = \left( x \cdot x - y \cdot y \right) - 2i \left( x \cdot y \right), \]
\[ \Rightarrow x \cdot x = y \cdot y, \quad \text{and} \quad x \cdot y = 0. \tag{18} \]

On the other hand, \( z \) is of norm 1 so that
\[ (z, z) = x \cdot x + y \cdot y = 1 \quad \Rightarrow \quad x \cdot x = y \cdot y = \frac{1}{2}. \]

Now we are ready to proceed with the exercise. First we show that there exists a complex number \( a \in \mathbb{C} \) such that \( v = az + \bar{a} \bar{z} \). Indeed, since \( z \) and \( \bar{z} \) span the space orthogonal to \( w \) in \( \mathbb{C}^3 \), there exist complex numbers \( a, b \in \mathbb{C} \) such that \( v = az + bz \). But then with \( x \) and \( y \) defined as in Proposition 3, the fact that \( v \) is a real vector implies
\[ (\text{Im}(a) + \text{Im}(b)) x + (\text{Re}(a) - \text{Re}(b)) y = 0 \Rightarrow b = \overline{a}, \tag{19} \]

where the implication is due to Proposition 3: \( x \) and \( y \) are orthogonal and of non zero norm so that both brackets must vanish independently.

Then the fact that \( v \) is a unit vector implies that \( |a|^2 = 1/2 \). Indeed, \( 1 = \langle v, v \rangle = \langle az + \overline{a} \overline{z}, az + \overline{a} \overline{z} \rangle = |a|^2 + |\overline{a}|^2 = 2|a|^2 \), where the third equality is due to Proposition 2.

Finally, simply compute
\[ \cos \theta = \frac{|a|^2(e^{-ix} + e^{ix})}{2} = \cos x. \tag{20} \]

4. Eigenvectors can always be multiplied by a non zero scalar and they remain eigenvectors. In particular the scalar can be complex. In the case of \( Q \in SO(3) \), this means that any real vector perpendicular to the axial vector \( u \) can be taken as the real part \( x \) of the eigenvector with complex eigenvalue \( e^{i\theta} \). The eigenvalue relation in terms of real and imaginary parts of the eigenvector \( z = x + iy \) becomes
\[ Qz = e^{i\theta}(x + iy) = (\cos(\theta)x - \sin(\theta)y) + i(\sin(\theta)x + \cos(\theta)y). \tag{21} \]

Then since \( Qz = Q(x + iy) \)
\[ Qx = \cos(\theta)x - \sin(\theta)y, \]
\[ Qy = \sin(\theta)x + \cos(\theta)y. \tag{22} \]

From proposition 3, we know that \( |y| = |x| \) & \( y \cdot x = 0 \). Thus, for real and imaginary parts of the eigenvectors we can take any \( x \) with \( x \cdot u = 0 \) and
\[ y = \pm(u \times x). \tag{23} \]

According to the right-hand rule if \( Q \) is a counter-clockwise rotation through the angle \( \phi \) about \( u \), and \( a = u \times v \)
\[ Qx = \cos(\phi)x + \sin(\phi)a, \]
\[ Qa = -\sin(\phi)x + \cos(\phi)a. \tag{24} \]
By comparing (22) with the relations according to the right-hand rule (eqn. 24), we find \( \phi = -\theta \) and \( y = + (u \times x) \) the choice of imaginary part of eigenvector for which \( \phi > 0 \) corresponds to an anti-clockwise direction about \( u \) and \( \{u, x, y\} \) is right-handed.

Now fix an axial vector \( u \) and consider an arbitrary vector \( v \in \mathbb{R}^3 \) which can be written as

\[
v = (u \cdot v)u + (I - u \otimes u)v. \tag{25}\]

Then we can consider \( x = (I - u \otimes u)v \) as the real part of the complex eigenvector and \( y = u \times ((I - u \otimes u)v) \) as the imaginary part. Then with \( \phi \) being a counter-clockwise relation about \( u \) we find

\[
Qv = (u \cdot v)Qu + Q(I - u \otimes u)v \tag{26}
\]
\[
= (u \cdot v)u + \cos(\phi)(I - u \otimes u)v + \sin(\phi)(u \times (I - u \otimes u)v) \tag{27}
\]
\[
= (1 - \cos(\phi))(u \otimes u)v + \sin(\phi)(u \times v) + \cos(\phi)v \tag{28}
\]

which gives (3) in the announce, because \( v \) is arbitrary. Then (2) follows using 1.3.