1 More properties of the Cayley transform

For what follows, \( P \in O(3) \) and \( Q \in SO(3) \).

1. Define \( \overline{\mathbf{u}} := Cay(Q^T) \), as before we remark that \( (Q^T \pm I) = (Q \pm I)^T \). Thus, we obtain
\[
\begin{align*}
[\overline{\mathbf{u}} \times] &= (Q + I)^{-T}(Q - I)^T = [(Q + I)^{-1}(Q - I)]^T = [u\times]^T = -[u\times].
\end{align*}
\]
(1)

We used the fact that \((Q + I)^{-1}\) and \((Q - I)\) commute. Finally we obtain that
\[
Cay(Q^T) = \overline{\mathbf{u}} = -u = -Cay(Q).
\]
(2)

2. Again define \( \overline{\mathbf{u}} := Cay(P^TQP) \), from the definition on the exercise sheet we obtain that
\[
[\overline{\mathbf{u}} \times] = (P^TQP + I)^{-1}(P^TQP - I)
\]
(3)

Since \( P^TP = I \) we have that \((P^TQP \pm I) = P^T(Q \pm I)P \). Moreover \((P^T(Q \pm I)P)^{-1} = P^T(Q \pm I)^{-1}P \), thus we obtain that
\[
[\overline{\mathbf{u}} \times] = P^T(Q + I)^{-1}(Q - I)P = P^T[u\times]P = [P|P^T u\times],
\]
(4)

where the last step uses Qn. 1.2 of Session 2. This implies
\[
Cay(P^TQP) = \overline{\mathbf{u}} = |P|P^Tu = |P|P^T Cay(Q).
\]
(5)

3. Again we can write
\[
Cay(Q) = Cay(RQR^T) = R [u\times] R^T \quad \text{(using property 2 from above)}
\]
\[
= [R| R u\times] \quad \text{(using Qn. 1.2 of Session 2)}
\]
\[
= R Cay(Q) \quad \text{(|R| = 1)}
\]
(6)

which implies \( \overline{\mathbf{u}} = Ru \).

2 The change of reading strand transformation

2.1 Proof of the change of reading strand transformation Part I

Let \((R, r^-)\) and \((R, r^+)\) be two frames and recall the following notion:

- the Cayley vector of the relative rotation from "minus" to "plus": \( u = Cay([-R^-]^T R^+) \in \mathbb{R}^3 \)
- the mid frame \((R, r)\): \( R = R^-([-R^-]^T R^+)^{\frac{1}{2}}, r = \frac{1}{2}(r^- + r^+) \)
- the relative translation: \( v = R^T(r^+ - r^-) \in \mathbb{R}^3 \)
Define now the analogous \( u \) and \( v \) but for the frames \((\overline{R}, \overline{r})\) by \( \overline{u} \) and \( \overline{v} \), and let \( P \in O(3) \). We want now to find the transformation between the Cayley vectors \( \overline{u} \) and \( u \):

\[
\overline{u} = \text{Cay}(\overline{R}^T \overline{R}) = \text{Cay}(P^T [R^+]^T R^- P) \quad (7)
\]

\[
= |P| P^T \text{Cay}([R^+]^T R^-) = |P| P^T \text{Cay}([[R^-]^T R^+]^T) \quad (8)
\]

\[
= -|P| P^T \text{Cay}([R^-]^T R^+) = -|P| P^T u \quad (9)
\]

Notice that we used the convention that the transformation is always from \(-\) to \(+\) but \( \overline{R} = R^+ P \) and \( \overline{R} = R^- P \). Before computing the transformation between the translational coordinates \( \overline{v} \) and \( v \) we observe that the mid frame \((R, r)\) can also be defined as the mid rotation from "plus" to "minus". In fact

\[
[R^- ([R^-]^T R^+]^{\frac{1}{2}}]^T R^+ ([R^+]^T R^-)^{\frac{1}{2}} = ([R^-]^T R^+)^{\frac{1}{2}} ([R^-]^T R^+)^{\frac{1}{2}} ([R^+]^T R^-)^{\frac{1}{2}}
\]

\[
= ([R^-]^T R^+)^{\frac{1}{2}} ([R^-]^T R^+)^{\frac{1}{2}}
\]

thus \( R = R^- ([R^-]^T R^+)^{\frac{1}{2}} = R^+ ([R^+]^T R^-)^{\frac{1}{2}} \). Now we can compute the mid frame \((\overline{R}, \overline{r})\):

\[
\overline{R} = \overline{R} ([\overline{R}^T \overline{R}]^{\frac{1}{2}}) = R^+ P (P^T [R^+]^T R^- P)^{\frac{1}{2}}
\]

\[
= R^+ ([R^+]^T R^-)^{\frac{1}{2}} P = RP
\]

Thus we can find the transformation from \( \overline{v} \) and \( v \):

\[
\overline{v} = \overline{R}^T (\overline{r}^+ - \overline{r}^-) = P^T R^T (r^+ - r^-) = -P^T R^T (r^+ - r^-) = -P^T v \quad (10)
\]

### 3 More on square roots of matrices

#### 3.1 On the general power of diagonalizable matrices

Let \( P \in O(3) \) and \( M \in \mathbb{R}^{3 \times 3} \) a diagonalizable matrix, with \( R \in \mathbb{R}^{3 \times 3} \) such that \( M = RDR^{-1} \), \( D = \text{diag}(\lambda_1, \lambda_2, \lambda_3) \).

We remark that \((P^T M P)^2 = (P^T M P)(P^T M P) = P^T MPP^T M P = P^T M^2 P \), because \( P^T P = I \). Thus we have that for any power \( \alpha \in \mathbb{N} \) we have

\[
(P^T M P)^\alpha = (P^T M P)(P^T M P)\ldots(P^T M P) = P^T M^\alpha P.
\]

Let \( \alpha \in \mathbb{R} \setminus \mathbb{N} \) and let us use the decomposition \( M = RDR^{-1} \). We obtain that \( P^T M P = P^T RDR^{-1} P \). Define \( \tilde{R} = P^T R \) thus, \( P^T M P = \tilde{R}D\tilde{R}^{-1} = \tilde{M} \). Now, \( M \in \mathbb{R}^{3 \times 3} \) is diagonalizable. Thus, we can apply the Definition 1 of the exercise sheet to obtain the generalized power \( \tilde{M}^\alpha \), i.e,

\[
(P^T M P)^\alpha = \tilde{M}^\alpha = \tilde{R}D^\alpha \tilde{R}^{-1} = P^T RD^\alpha R^{-1} P = P^T M^\alpha P. \quad (11)
\]

The statement is proven.

#### 3.2 Square roots in SE(3)

Using a direct computation one can easily find that

\[
B = \begin{bmatrix}
Q^{\frac{1}{2}} & (I + Q^{\frac{1}{2}})^{-1} q
\end{bmatrix}.
\]

The transformation rules for this definition of mid-frame are much more complicated than those in (1-2) for the other choice of mid-frame.