1 The length scales of DNA

Every diploid human cell, in the non-dividing state, contains twice the human genome ($\approx 3 \cdot 10^9$ base-pairs of DNA) for a total of about $6 \cdot 10^9$ base-pairs of DNA. The diameter of a typical human nucleus cell is of the order of $10^5 \mu m = 10^5 \text{Å}$, where $1 \mu m = 10^{-6} \text{m}$, $1 \text{nm} = 10^{-9} \text{m}$, $1 \text{Å} = 10^{-10} \text{m}$.

1. By treating the DNA as a cylinder of length $3.4 \text{Å}$ per base-pair, calculate the total length of DNA in any cell. Compare this to the diameter of the cell.

2. By treating the DNA as a cylinder of diameter $20 \text{Å}$, calculate the total volume of DNA in any cell. Compare this to the total volume of the cell, if it is assumed to be spherical.

Remark: This exercise was taken from the book *Understanding DNA*, C.R. Calladine, H.R. Drew

2 Gaussian integrals I

Let $\beta > 0$, $n > 0$, $\hat{x} \in \mathbb{R}^n$, and a symmetric, positive-definite matrix $K \in \mathbb{R}^{n \times n}$ be given. Show that

1. The one-dimensional Gaussian integral is
   \[
   \int_{-\infty}^{\infty} e^{-\sigma^2} d\sigma = \sqrt{\pi}.
   \]

2. The n-dimensional Gaussian integral is
   \[
   Z^{(n)} := \int_{\mathbb{R}^n} e^{-\beta(x-\hat{x}) \cdot K(x-\hat{x})} dx = \int_{\mathbb{R}^n} e^{-\beta x \cdot K x} dx = \left( \frac{\pi}{\beta} \right)^{\frac{n}{2}} \sqrt{\det[K^{-1}]}.
   \]

3. \[
   \frac{1}{Z^{(n)}} \int_{\mathbb{R}^n} x_i e^{-\beta x \cdot K x} dx = 0,
   \]
   \[
   \frac{1}{Z^{(n)}} \int_{\mathbb{R}^n} x_i e^{-\beta(x-\hat{x}) \cdot K(x-\hat{x})} dx = \hat{x}_i, \quad 1 \leq i \leq n.
   \]

4. \[
   \frac{1}{Z^{(n)}} \int_{\mathbb{R}^n} (x_i - \hat{x}_i)(x_j - \hat{x}_j) e^{-\beta(x-\hat{x}) \cdot K(x-\hat{x})} dx
   = \frac{1}{Z^{(n)}} \int_{\mathbb{R}^n} x_i x_j e^{-\beta x \cdot K x} dx
   = \frac{1}{2\beta} K^{-1}_{ij}, \quad (1 \leq i, j \leq n).
   \]

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1 A diploid cell contains two copies of each chromosome, plus a couple X X for women and a couple X Y for men.
5. Define the following quadratic form $U(x) = \frac{1}{2}(x - \hat{x}) \cdot K(x - \hat{x})$. Compute the expectation of $U(x)$ with respect to $p(x) = \frac{1}{Z} \exp(-\beta U(x))$, i.e:

$$\langle U(x) \rangle_p = \int_{\mathbb{R}^N} U(x)p(x)dx.$$ 

Remark: In the DNA context $\beta = \frac{1}{k_BT}$, where $k_B$ is the Boltzmann constant and $T$ is the kinetic temperature of the bath where the DNA is immersed in. For convenience, in this lecture, we will often use $\beta = 1$ and we will absorb $k_BT$ into the stiffness matrix $K$.

[Hint: For 1 calculate $\left( \int_{\mathbb{R}} e^{-\sigma^2}d\sigma \right)^2$. Then use 1 to solve 2-4.]