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| **Mathematical Modelling of DNA** |
| PROF. JOHN MADDOCKS — VERSION SPRING 2021 |
| SESSION 1 — A. PATELLI, R. SINGH |

# 1 The length scales of DNA

Every diploid human cell nucleus[1], in the non-dividing state, contains twice the human genome ($\simeq 3 \cdot 10^9$ base-pairs of DNA) for a total of about $6 \cdot 10^9$ base-pairs of DNA. The diameter of a typical human cell nucleus is of the order of $10\mu\text{m} = 10^5$Å, where

$$\mu\text{m} = 10^{-6}\text{m}, \qquad \text{nm} = 10^{-9}\text{m}, \quad \text{Å} = 10^{-10}\text{m}.$$

**1.** By treating the DNA as a cylinder of length 3.4Å per base-pair, calculate the total length of DNA in any typical human cell nucleus. Compare this to the diameter of nucleus.

**2.** By treating the DNA as a cylinder of diameter 20Å, calculate the total volume of DNA in any typical human cell nucleus. Compare this to the total volume of the cell nucleus, if it is assumed to be spherical.

**3.** Compute the aspect ratio (i.e. total arc length divided by diameter) of the total DNA in each nucleus. To get a better sense of what this means multiply the scales by $10^6$ so that the DNA diameter would be approximately 2mm instead of 2nm (2mm is about the width of a typical chalk line drawn on the blackboard). At this scale how far would all the DNA stretch if it were imagined to all be laid out in one straight line (which of course it is not).

**Remark**: This exercise was taken from the book *Understanding DNA*, C.R. Calladine, H.R. Drew.

# 2 Gaussian integrals I

Let $\beta > 0$, $n > 0$, $\hat{x} \in \mathbb{R}^n$ , and a symmetric, positive - definite matrix $K \in \mathbb{R}^{n\times n}$ be given. Show that

**1.** The one-dimensional Gaussian integral is

$$\int_{-\infty}^{\infty} e^{-\frac{1}{2}\sigma^2} d\sigma = \sqrt{2\pi}.$$

**2.** The n-dimensional Gaussian integral is

$$Z^{(n)} := \int_{\mathbb{R}^n} e^{-\frac{\beta}{2}(x-\hat{x})\cdot K(x-\hat{x})} dx = \int_{\mathbb{R}^n} e^{-\frac{\beta}{2}x\cdot Kx} dx = \left(\frac{2\pi}{\beta}\right)^{\frac{n}{2}} \sqrt{\det[K^{-1}]}.$$

**3.**

$$\begin{aligned} \frac{1}{Z^{(n)}} \int_{\mathbb{R}^n} x_i e^{-\frac{\beta}{2}x\cdot Kx}\, dx &= 0, \\ \frac{1}{Z^{(n)}} \int_{\mathbb{R}^n} x_i e^{-\frac{\beta}{2}(x-\hat{x})\cdot K(x-\hat{x})}\, dx &= \hat{x}_i, \quad 1 \le i \le n. \end{aligned}$$

[1] A diploid human cell nucleus contains two copies of each chromosome, plus a couple X X for women and a couple X Y for men.

**4.**

$$
\begin{aligned}
& \frac{1}{Z^{(n)}} \int_{\mathbb{R}^n} (x_i - \hat{x}_i)(x_j - \hat{x}_j)\, e^{-\frac{\beta}{2}(x-\hat{x})\cdot K(x-\hat{x})}\, dx \\
= \;& \frac{1}{Z^{(n)}} \int_{\mathbb{R}^n} x_i x_j e^{-\frac{\beta}{2} x\cdot Kx}\, dx \\
= \;& \frac{1}{\beta} K^{-1}_{ij}, \quad (1 \leq i, j \leq n).
\end{aligned}
$$

**5.** Define the following quadratic form $U(x) = \frac{1}{2}(x - \hat{x}) \cdot K(x - \hat{x})$. Compute the expectation of $U(x)$ with respect to $p(x) = \frac{1}{Z} exp(-\beta U(x))$, i.e :

$$
\langle U(x) \rangle_p = \int_{\mathbb{R}^N} U(x) p(x) dx.
$$

**Remark:** In the DNA context $\beta = \frac{1}{k_B T}$, where $k_B$ is the Boltzmann constant and $T$ is the kinetic temperature of the bath where the DNA is immersed in. For convenience, in this lecture, we will often use $\beta = 1$ and we will absorb $k_B T$ into the stiffness matrix $K$.

[Hint: For 1 calculate $\left( \int_{-\infty}^{\infty} e^{-\sigma^2} d\sigma \right)^2$. Then use 1 to solve 2-4.]