Mathematical Modelling of DNA
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SESSION 1 A. PATELLI

1 The length scales of DNA

Every diploid human cell[1], in the non-dividing state, contains twice the human genome ($\simeq 3 \cdot 10^9$ base-pairs of DNA) for a total of about $6 \cdot 10^9$ base-pairs of DNA. The diameter of a typical human nucleus cell is of the order of $10\mu\text{m} = 10^5\text{Å}$, where

$$\mu\text{m} = 10^{-6}\text{m}, \qquad \text{nm} = 10^{-9}\text{m}, \quad \text{Å} = 10^{-10}\text{m}.$$

1. By treating the DNA as a cylinder of length 3.4Å per base-pair, calculate the total length of DNA in any cell. Compare this to the diameter of the cell.

2. By treating the DNA as a cylinder of diameter 20Å, calculate the total volume of DNA in any cell. Compare this to the total volume of the cell, if it is assumed to be spherical.

Remark: This exercise was taken from the book *Understanding DNA*, C.R. Calladine, H.R. Drew

2 Gaussian integrals I

Let $\beta > 0$, $n > 0$, $\hat{x} \in \mathbb{R}^n$, and a symmetric, positive - definite matrix $K \in \mathbb{R}^{n \times n}$ be given. Show that

1. The one-dimensional Gaussian integral is

$$\int_{-\infty}^{\infty} e^{-\sigma^2} d\sigma = \sqrt{\pi}.$$

2. The n-dimensional Gaussian integral is

$$Z^{(n)} := \int_{\mathbb{R}^n} e^{-\beta(x-\hat{x})\cdot K(x-\hat{x})} dx = \int_{\mathbb{R}^n} e^{-\beta x \cdot Kx} dx = \left(\frac{\pi}{\beta}\right)^{\frac{n}{2}} \sqrt{\det[K^{-1}]}.$$

3.

$$\begin{aligned} \frac{1}{Z^{(n)}} \int_{\mathbb{R}^n} x_i e^{-\beta x \cdot Kx}\, dx &= 0, \\ \frac{1}{Z^{(n)}} \int_{\mathbb{R}^n} x_i e^{-\beta(x-\hat{x})\cdot K(x-\hat{x})}\, dx &= \hat{x}_i, \quad 1 \leq i \leq n. \end{aligned}$$

4.

$$\begin{aligned} & \frac{1}{Z^{(n)}} \int_{\mathbb{R}^n} (x_i - \hat{x}_i)(x_j - \hat{x}_j)\, e^{-\beta(x-\hat{x})\cdot K(x-\hat{x})}\, dx \\ = \; & \frac{1}{Z^{(n)}} \int_{\mathbb{R}^n} x_i x_j e^{-\beta x \cdot Kx}\, dx \\ = \; & \frac{1}{2\beta} K_{ij}^{-1}, \quad (1 \leq i, j \leq n). \end{aligned}$$

[1]A diploid cell contains two copies of each chromosome, plus a couple X X for women and a couple X Y for men.

5. Define the following quadratic form $U(x) = \frac{1}{2}(x - \hat{x}) \cdot K(x - \hat{x})$. Compute the expectation of $U(x)$ with respect to $p(x) = \frac{1}{Z} exp(-\beta U(x))$, i.e :

$$\langle U(x) \rangle_p = \int_{\mathbb{R}^N} U(x) p(x) dx.$$

Remark: In the DNA context $\beta = \frac{1}{k_B T}$, where k_B is the Boltzmann constant and T is the kinetic temperature of the bath where the DNA is immersed in. For convenience, in this lecture, we will often use $\beta = 1$ and we will absorb $k_B T$ into the stiffness matrix K.

[Hint: For 1 calculate $\left(\int_{-\infty}^{\infty} e^{-\sigma^2} d\sigma \right)^2$. Then use 1 to solve 2-4.]