1 Properties of skew symmetric matrices

1. Given an orthonormal basis \{e_1, e_2, e_3\} and a particular vector \(u\) in \(\mathbb{R}^3\), write the skew symmetric matrix \([u \times]\) that applied to any vector \(v\) gives \(u \times v\). Verify that the resulting mapping

\[
\mathbb{R}^3 \rightarrow \mathbb{R}^{3 \times 3} \\
u \mapsto [u \times]
\]

is linear and invertible.

2. Show that for any invertible matrix \(M \in \mathbb{R}^{3 \times 3}\),

\[
[M u \times] = |M| M^{-T} [u \times] M^{-1}.
\]

How does this formula simplify if \(M \in SO(3)\)? The relation you will obtain is the change of basis formula. Notation: \([a \times]\) assumes that \(a\) is a triple of numbers. Thus \([Ma \times]\) for \(M \in \mathbb{R}^{3 \times 3}\) means explicitly \([(Ma) \times] (a \in \mathbb{R}^3, M \in \mathbb{R}^{3 \times 3}, Ma \in \mathbb{R}^3)\) but we usually drop the parenthesis so \([(Ma)\times] \equiv [Ma\times]\)

3. Prove that, for any \(u \in \mathbb{R}^3\), we have

\[
[u \times]^2 = u \otimes u - |u|^2 I
\]

[Hint: Distribute the triple vector product \(u \times (u \times v)\) for an arbitrary vector \(v\)]

4. Compute the characteristic polynomial for the matrix \([u \times]\) and thereby show that the eigenvalues of \([u \times]\) are \(0, \pm |u|\). Using 3 (or directly) verify that \([u \times]\) satisfies its characteristic polynomial (It must, by the Cayley-Hamilton theorem, but we verify directly as we shall use it later).

2 Rotations in three dimensions

Consider any matrix \(Q \in SO(3)\).

1. Show that all the eigenvalues of \(Q\) are on the unit circle in the complex plane.

2. Show that \(Q\) always has an eigenvalue of unity, so that there is a unit vector \(w\) such that \(Qw = w\). This vector defines the axis of rotation of \(Q\) and is parallel to the axial vector of the skew matrix \(Q - Q^T\). Can a proper rotation have more than one axis?

3. Let \(v\) be any unit vector orthogonal to \(w\). Show that \(Qv\) is also a unit vector orthogonal to \(w\) and that the angle \(0 \leq \theta \leq \pi\) between \(v\) and \(Qv\) satisfies the relation

\[
1 + 2 \cos \theta = \text{tr}(Q).
\]

[Hint: Express \(\text{tr}(Q)\) in terms of the eigenvalues.]
4. Let \( Q \in SO(3) \) be a rotation matrix with respect to the basis \( \{e_1, e_2, e_3\} \) and denote by \( u \) its unit rotation axis and by \( \phi \) its rotation angle. Prove that \( Q \) can be written as

\[
Q = I + \sin \phi [u \times] + (1 - \cos \phi)[u \times]^2. \tag{2}
\]

Moreover, show that (2) can also be expressed as

\[
Q = \cos \phi I + \sin \phi [u \times] + (1 - \cos \phi)u \otimes u. \tag{3}
\]

[Hint: For (3) \( \iff \) (2) use exercise 1.3]

Relations (2) and (3) are two matrix forms of so-called Euler-Rodrigues formulas, which give an explicit expression for rotation matrix \( Q \) about an axis \( u \) through an angle \( \phi \).