1 Properties of skew symmetric matrices

1. Given an orthonormal basis \( \{e_1, e_2, e_3\} \) and a particular vector \( u \) in \( \mathbb{R}^3 \), write the skew symmetric matrix \([u \times]\) that applied to any vector \( v \) gives \( u \times v \). Verify that the resulting mapping

\[
\mathbb{R}^3 \rightarrow \mathbb{R}^{3 \times 3} \\
\mathbf{u} \mapsto [\mathbf{u} \times]
\]

is linear and invertible.

2. Show that for any invertible matrix \( M \in \mathbb{R}^{3 \times 3} \),

\[
[M \mathbf{u} \times] = |M| M^{-T} [\mathbf{u} \times] M^{-1}.
\]

How does this formula simplify if \( M \in SO(3) \)? The relation you will obtain is the change of basis formula. Notation: \([a \times]\) assumes that \( a \) is a triple of numbers. Thus \([Ma \times]\) for \( M \in \mathbb{R}^{3 \times 3} \) means explicitly \([(Ma) \times] (a \in \mathbb{R}^3, M \in \mathbb{R}^{3 \times 3}, Ma \in \mathbb{R}^3)\) but we usually drop the parenthesis so \([(Ma) \times] \equiv [Ma \times]\)

3. Prove that, for any \( \mathbf{u} \in \mathbb{R}^3 \), we have

\[
[u \times]^2 = u \otimes u - |u|^2 I
\]

[Hint: Distribute the triple vector product \( \mathbf{u} \times (\mathbf{u} \times \mathbf{v}) \) for an arbitrary vector \( \mathbf{v} \)]

4. Compute the characteristic polynomial for the matrix \([u \times]\) and thereby show that the eigenvalues of \([u \times]\) are \(0, \pm |u|\). Using 3 (or directly) verify that \([u \times]\) satisfies its characteristic polynomial (It must, by the Cayley-Hamilton theorem, but we verify directly as we shall use it later).

2 Rotations in three dimensions

Consider any matrix \( Q \in SO(3) \).

1. Show that all the eigenvalues of \( Q \) are on the unit circle in the complex plane.

2. Show that \( Q \) always has an eigenvalue of unity, so that there is a unit vector \( \mathbf{w} \) such that \( Q \mathbf{w} = \mathbf{w} \). This vector defines the axis of rotation of \( Q \) and is parallel to the axial vector of the skew matrix \( Q - Q^T \). Can a proper rotation have more than one axis?

3. Let \( \mathbf{v} \) be any unit vector orthogonal to \( \mathbf{w} \). Show that \( Q \mathbf{v} \) is also a unit vector orthogonal to \( \mathbf{w} \) and that the angle \( 0 \leq \theta \leq \pi \) between \( \mathbf{v} \) and \( Q \mathbf{v} \) satisfies the relation

\[
1 + 2 \cos \theta = \text{tr}(Q) \tag{1}
\]

[Hint: Express \( \text{tr}(Q) \) in terms of the eigenvalues.]
4. Let $Q \in SO(3)$ be a rotation matrix with respect to the basis $\{e_1, e_2, e_3\}$ and denote by $u$ its unit rotation axis and by $\phi$ its rotation angle.

Prove that $Q$ can be written as

$$Q = I + \sin \phi [u \times] + (1 - \cos \phi) [u \times]^2.$$  \hspace{1cm} (2)

Moreover, show that (2) can also be expressed as

$$Q = \cos \phi I + \sin \phi [u \times] + (1 - \cos \phi) u \otimes u.$$ \hspace{1cm} (3)

[Hint: For (3) \iff (2) use exercise 1.3]

Relations (2) and (3) are two matrix forms of so-called Euler-Rodrigues formulas, which give an explicit expression for rotation matrix $Q$ about an axis $u$ through an angle $\phi$. 