1 Properties of skew symmetric matrices

1. Given an orthonormal basis $\{\mathbf{e}_1, \mathbf{e}_2, \mathbf{e}_3\}$ and a particular vector $\mathbf{u}$ in $\mathbb{R}^3$, write the skew symmetric matrix $[\mathbf{u}\times]$ that applied to any vector $\mathbf{v}$ gives $\mathbf{u} \times \mathbf{v}$. Verify that the resulting mapping

$$\mathbb{R}^3 \longmapsto \mathbb{S}^{3\times 3} \subset \mathbb{R}^{3\times 3}, \qquad \mathbb{S}^{3\times 3} = \{S = \mathbb{R}^{3\times 3} : S = -S^T\}$$
$$\mathbf{u} \longmapsto [\mathbf{u}\times]$$

is linear and invertible. We call $\mathbf{u}$ the axial vector of the skew symmetric matrix $S = [\mathbf{u}\times]$.

2. Show that for any invertible matrix $M \in \mathbb{R}^3 \times \mathbb{R}^3$,

$$[M\mathbf{u}\times] = |M|\, M^{-T}\, [\mathbf{u}\times]\, M^{-1}.$$

How does this formula simplify if $M \in \mathrm{SO}(3)$? The relation you will obtain is the change of basis formula.

Notation: $[a\times]$ asssumes that a is a triple of numbers. Thus $[Ma\times]$ for $M \in \mathbb{R}^{3\times 3}$ means explicitly $[(Ma)\times]$ ($a \in \mathbb{R}^3$, $M \in \mathbb{R}^{3\times 3}$, $Ma \in \mathbb{R}^3$) but we usually drop the parenthesis so $[(Ma)\times] \equiv [Ma\times]$. And $|M|$ means determinant of any matrix M which can of course be positive negative or zero.

3. Prove that, for any $\mathbf{u} \in \mathbb{R}^3$, we have

$$[\mathbf{u}\times]^2 = \mathbf{u} \otimes \mathbf{u} - |\mathbf{u}|^2 \mathrm{I}$$

[Hint: Distribute the triple vector product $\mathbf{u} \times (\mathbf{u} \times \mathbf{v})$ for an arbitrary vector $\mathbf{v}$]

4. Compute the characteristic polynomial for the matrix $[\mathbf{u}\times]$ and thereby show that the eigen values of $[\mathbf{u}\times]$ are $0, \pm i|\mathbf{u}|$. Using **3** (or directly) verify that $[\mathbf{u}\times]$ satisfies its characteristic polynomial (It must, by the Cayley-Hamilton theorem, but we verify directly as we shall use it later).

2 Rotations in three dimensions

Consider any matrix $Q \in SO(3)$ i.e. Q is 3×3 with $|Q| = +1$ and $Q^T = Q^{-1}$, so that $QQ^T = Q^T Q = \mathrm{I}$.

1. Show that all the eigenvalues of Q are on the unit circle in the complex plane.

2. Show that Q always has an eigenvalue of unity, so that there is a unit vector $\mathbf{w}$ such that $Q\mathbf{w} = \mathbf{w}$. We will show that $\mathbf{w}$ defines the axis of rotation of Q and is parallel (or anti parallel) to the axial vector $\mathbf{u}$ of the skew matrix $Q - Q^T$. Can a proper rotation have more than one axis?

3. Let $\mathbf{v}$ be a vector orthogonal to $\mathbf{w}$. Show that $Q\mathbf{v}$ is also a vector orthogonal to $\mathbf{w}$ and that the angle $0 \leq \theta \leq \pi$ between $\mathbf{v}$ and $Q\mathbf{v}$ satisfies the relation

$$1 + 2\cos\theta = \mathrm{tr}(Q). \tag{1}$$

[Hint: Express $\mathrm{tr}(Q)$ in terms of the eigenvalues.]

4. Using the convention that $Q \in SO(3)$ is a rotation about a unit vector $\mathbf{n} \in \mathbb{R}^3$ (called the unit rotation axis) in the counter clock wise sense (from right handed rule) of angle $\theta \in [0,\ \pi]$, what is the relation between the eigenvector $\mathbf{w}$ of part 1, the axial vector $\mathbf{u}$ of the skew matrix $Q - Q^T$ of part 2, and the unit rotation axis $\mathbf{n}$?

5. Let $Q \in SO(3)$ be a rotation matrix with respect to the basis $\{\mathbf{e}_1, \mathbf{e}_2, \mathbf{e}_3\}$ and denote by $\mathbf{n}$ its unit rotation axis and by ϕ its rotation angle.
Prove that Q can be written as

$$Q = \mathrm{I} + \sin\phi\,[\mathbf{n}\times] + (1 - \cos\phi)\,[\mathbf{n}\times]^2. \tag{2}$$

Moreover, show that (2) can also be expressed as

$$Q = \cos\phi\,\mathrm{I} + \sin\phi\,[\mathbf{n}\times] + (1 - \cos\phi)\,\mathbf{n} \otimes \mathbf{n}. \tag{3}$$

3 Computing matrix exponential using Cayley-Hamilton

Use the Taylor series definition of the matrix exponential

$$exp(M) = I + M + \frac{M^2}{2!} + \frac{M^3}{3!} + \cdots = \sum_{n=0}^{\infty} \frac{1}{n!} M^n \tag{4}$$

applied with $M = [\mathbf{u}\times]$ & the result of 1.5 to show (the form (3) of question 2)

$$exp(M) \quad = \quad \mathrm{I} + \sin\phi[\mathbf{n}\times] + (1 - \cos\phi)\,[\mathbf{n}\times]^2 \tag{5}$$

where $\phi = |\mathbf{u}|$ and $\mathbf{n} = \frac{\mathbf{u}}{|\mathbf{u}|}$. Thus we see for any fix S skew-symmetric $exp(S) \in SO(3)$.