1 Cayley transforms

Let $N \in \mathbb{R}^{n \times n}$ such that $|I - N| \neq 0$. The Cayley transform of $N$ is the matrix $M$ defined by

$$M = (I + N)(I - N)^{-1},$$

where $I$ is the identity matrix in $\mathbb{R}^{n \times n}$.

1.1 A few general properties

1. Show that $(I + N)(I - N)^{-1} = (I - N)^{-1}(I + N)$.

2. Show that if $M$ is the Cayley transform of some matrix $N$, then the matrix $I + M$ is invertible. [Hint: Use 1.1.1 and use that $I + M$ is not invertible iff $\exists v \neq 0 : (I + M)v = 0$.]

3. Show that if $M$ is the Cayley transform of some matrix $N$, then the inverse Cayley transform of $M$ is

$$N = (M + I)^{-1}M^{-1}(M - I) = -(I - M)(I + M)^{-1}.$$  

4. Replace $N = -P$ in (1), which is just an alternative sign convention leading an alternative definition of the Cayley transform $M$ of the matrix $P$:

$$M = (I - P)(I + P)^{-1}.$$ 

Show that with this convention the inverse transform coincides with the transform, i.e.

$$P = (I - M)(I + M)^{-1}.$$ 

(which is why some authors use this convention, but we will stick with (1) for reasons explained later.)

5. Assume that $Q$ is the Cayley transform of $S$. Show that $Q \in SO(n)$ if and only if $S$ is skew. Is it true that for a general matrix $R \in SO(n)$ there exists a skew matrix $U$ such that $R$ is the Cayley transform of $U$?

1.2 The case of $SO(3)$

Assume that $Q$ is the Cayley transform of a skew-symmetric matrix $S = [u \times] \in \mathbb{R}^{3 \times 3}$. Note that vector $u$ is not unit vector which is in contrast to vector $n$ appearing in question 2 of Session 2.

1. Show that

$$Q = \frac{1 - \|u\|^2}{1 + \|u\|^2}I + \frac{2}{1 + \|u\|^2}[u \times] + \frac{2}{1 + \|u\|^2}u \otimes u.$$  

Equation (2) is another version of the Euler–Rodrigues formulas (2) and (3) seen in exercise 2, session 2.

Interpreting $Q$ as a rotation matrix, show that it corresponds to a right-handed rotation by an angle $2 \arctan(\|u\|) \in [0, \pi)$ about the unit axis vector $n = +u/\|u\|$.

2. Provided that $\text{tr} Q \neq -1$ (i.e. $Q$ is not a rotation through $\pi$). Show that

$$S \equiv [u \times] = (Q - Q^T)/(1 + \text{tr} Q).$$
Note that equations (2) & (3) give explicit forms of the mapping and its inverse between vectors \( u \in \mathbb{R}^3 \) & \( \text{SO}(3) \) rotation matrices through an angle \([0, \pi]\) (for which \((1 + \text{tr} Q) > 0\)).

1.3 The case of \( SE(3) \)

We recall that an element \( R \in SE(3) \) has the following matrix form

\[
R = \begin{bmatrix}
R & r \\
0^T & 1
\end{bmatrix} \in \mathbb{R}^{4 \times 4},
\]

where \( R \in \text{SO}(3) \) and \( r \in \mathbb{R}^3 \).

Show that if \( Q \) is the Cayley transform of \( S \in \mathbb{R}^{4 \times 4} \), then \( Q \in SE(3) \) if and only if there exists two vectors \( u \) and \( v \) such that

\[
S = \begin{bmatrix}
[u \times] & v \\
0 & 0
\end{bmatrix}.
\]