# 1 Cayley transforms

Let $N \in \mathbb{R}^{n\times n}$ such that $|I - N| \neq 0$. The Cayley transform of $N$ is the matrix $M$ defined by

$$M = (\mathrm{I} + N)(\mathrm{I} - N)^{-1}, \tag{1}$$

where $I$ is the identity matrix in $\mathbb{R}^{n\times n}$.

## 1.1 A few general properties

**1.** Show that $(\mathrm{I} + N)(\mathrm{I} - N)^{-1} = (\mathrm{I} - N)^{-1}(\mathrm{I} + N)$.

**2.** Show that if $M$ is the Cayley transform of some matrix $N$, then the matrix $\mathrm{I}+M$ is invertible. [Hint: Use 1.1.1 and use that $\mathrm{I} + M$ is not invertible iff $\exists \mathbf{v} \neq \mathbf{0} : (\mathrm{I} + M)\mathbf{v} = \mathbf{0}$.]

**3.** Show that if $M$ is the Cayley transform of some matrix $N$, then the inverse Cayley transform of $M$ is $N = (M + \mathrm{I})^{-1}(M - \mathrm{I}) = -(\mathrm{I} - M)(\mathrm{I} + M)^{-1}$.

**4.** Replace $N = -P$ in (1), which is just an alternative sign convention leading an alternative definition of the Cayley transform $M$ of the matrix $P$:

$$M = (\mathrm{I} - P)(\mathrm{I} + P)^{-1}.$$

Show that with this convention the inverse transform coincides with the transform, i.e.

$$P = (\mathrm{I} - M)(\mathrm{I} + M)^{-1}.$$

(which is why some authors use this convention, but we will stick with (1) for reasons explained later.)

**5.** Assume that $Q$ is the Cayley transform of $S$. Show that $Q \in SO(n)$ if and only if $S$ is skew. Is it true that for a general matrix $R \in SO(n)$ there exists a skew matrix $U$ such that $R$ is the Cayley transform of $U$?

## 1.2 The case of $SO(3)$

Assume that $Q$ is the Cayley transform of a skew-symmetric matrix $S = [\mathbf{u}\times] \in \mathbb{R}^{3\times 3}$. Note that vector $\mathbf{u}$ is not unit vector which is in contrast to vector $\mathbf{n}$ appearing in question 2 of Session 2.

**1.** Show that

$$Q = \frac{1 - \|\mathbf{u}\|^2}{1 + \|\mathbf{u}\|^2}\,\mathrm{I} + \frac{2}{1 + \|\mathbf{u}\|^2}[\mathbf{u}\times] + \frac{2}{1 + \|\mathbf{u}\|^2}\mathbf{u} \otimes \mathbf{u}. \tag{2}$$

Equation (2) is another version of the Euler–Rodrigues formulas (2) and (3) seen in exercise 2, session 2.
Interpreting $Q$ as a rotation matrix, show that it corresponds to a right-handed rotation by an angle $2\,\mathrm{Arctan}(\|\mathbf{u}\|) \in [0, \pi)$ about the unit axis vector $\mathbf{n} = +\mathbf{u}/\|\mathbf{u}\|$.

**2.** Provided that $\mathrm{tr}\, Q \neq -1$ (i.e. $Q$ is not a rotation through $\pi$). Show that

$$S \equiv [\mathbf{u}\times] = (Q - Q^T)/(1 + \mathrm{tr}\, Q). \tag{3}$$

Note that equations (2) & (3) give explicit forms of the mapping and its inverse between vectors $\mathbf{u} \in \mathbb{R}^3$ & $SO(3)$ rotation matrices through an angle $[0, \pi)$ (for which $(1 + \operatorname{tr} Q) > 0$).

### 1.3 The case of $SE(3)$

We recall that an element $\mathcal{R} \in SE(3)$ has the following matrix form

$$\mathcal{R} = \begin{bmatrix} R & \mathbf{r} \\ \mathbf{0}^T & 1 \end{bmatrix} \in \mathbb{R}^{4\times 4},$$

where $R \in SO(3)$ and $\mathbf{r} \in \mathbb{R}^3$.

Show that if $\mathcal{Q}$ is the Cayley transform of $\mathcal{S} \in \mathbb{R}^{4\times 4}$, then $\mathcal{Q} \in SE(3)$ if and only if there exists two vectors $\mathbf{u}$ and $\mathbf{v}$ such that

$$\mathcal{S} = \begin{pmatrix} [\mathbf{u}\times] & \mathbf{v} \\ 0 & 0 \end{pmatrix}.$$