Review of Matrix Factorisation (or should be a review depending on what Linear Algebra courses you have followed, or wikipedia is your friend)

In the course and exercises, we will use various matrix factorisation of a matrix M that is a) square and b) has real entries. Most of the factorisations apply for more general M, and for complex entries, but we will not use those cases in the course.

$$M = LU \tag{1}$$

where L is unit lower triangular (i.e. lower triangular and all diagonal entries are equal to one) and U is upper triangular. Most basic factorisation used in Gaussian elimination and solving linear systems. When M is symmetric, it specializes to

$$M = LDL^T \tag{2}$$

with D a diagonal matrix. And when M is symmetric positive definite each $d_{ii} > 0$. This leads to the Cholesky factorisation of a symmetric positive definite matrix

$$M = KK^T, \tag{3}$$

with $K := L\sqrt{D}$ lower triangular. When M is banded and symmetric both L, and therefore K, are also banded, which is important for our Monte Carlo code to be seen later in the course.

For M symmetric, there is also the spectral decomposition which arises because M has n real eigenvalues (counting algebraic multiplicity) and n real eigenvectors (i.e. geometric and algebraic multiplicities are equal) such that the eigenvectors ξ_i can be chosen orthonormal, $\xi_i \cdot \xi_j = \delta_{ij}$. The eigenvalue relations then become

$$MP = P\Lambda \tag{4}$$

where the columns of P are the eigenvectors ξ_i and Λ is diagonal with diagonal entries the eigenvalues λ_i of M. Orthonormality of the ξ_i means that the matrix $P \in O(n)$ i.e. $P^{-1} = P^T$ (and by taking $\pm\xi_i$ can always generate $P \in SO(n)$ i.e. $\det P = +1$). The eigenvalue equation (4) then immediately gives the spectral decomposition

$$M = P\Lambda P^T \tag{5}$$

When M is positive definite $\lambda_i > 0$, so

$$M = \tilde{K}\tilde{K}^T \tag{6}$$

where $\tilde{K} = P\sqrt{\Lambda}$. However in general, and even if M is banded, the matrix P with eigenvectors as columns is dense, so in contrast to the Cholesky factor K, the matrix $\tilde{K}$ is also dense and not even lower triangular, never mind banded.

For *normal* matrices (i.e. $\exists$ n linearly independent eigenvectors) the spectral decomposition takes the form

$$M = P\Lambda P^{-1} \tag{7}$$

where now the matrix P of eigenvectors is no longer orthonormal (but is still invertible by the assumption of normality).

We will use another generalisation for any non-symmetric matrix (normal or not) - the Singular Value Decomposition or SVD. (The SVD is also often used for non-square matrices, but we will only use the version for square, but generally non-symmetric matrices). Both MM^T and M^TM are square symmetric positive semi-definite matrices (which are positive definite if M is of full rank). MM^T and M^TM are generally different matrices with different matrices of eigenvectors, $U \in O(n)$ for MM^T and $V \in O(n)$ for M^TM. However, one can easily show that MM^T and M^TM have the same eigenvalues $\lambda_i \geq 0$, $i = 1, 2, ..., n$. The singular values of M are $\sigma_i := \sqrt{\lambda_i} \geq 0$, and the SVD of M is

$$M = U\Sigma V^T, \Sigma = \text{Diag}\{\sigma_i\} \tag{8}$$

(so that $MV = U\Sigma$ and $U^TM = \Sigma V^T$ and $MM^TU = U\Sigma^2$, $M^TMV = V\Sigma^2$). Usual convention is to order the singular values σ_i in decreasing order, which is unique in the case that all the σ_i are distinct. Consequently with this convention it is possible that either $\det(U)$ or $\det(V)$ are negative, so that while U and V are orthogonal they may not be proper orthogonal.

Finally, we will use the (least standard) polar decomposition factorisation of M,

$$M = WP \tag{9}$$

with $W \in O(n)$, $WW^T = Id$ and $P = P^T \geq 0$. The factorisation always exists, and is unique if M is full rank/invertible in which case $P > 0$, then $W \in SO(n)$ if and only if $\det M > 0$. The polar decomposition can be proven directly, but it follows immediately if you know the SVD.

$$M = U\Sigma V^T = UV^TV\Sigma V^T = WP, \tag{10}$$

where $UV^T = W \Rightarrow W^TW = I_d$ and $V\Sigma V^T = P \Rightarrow P = P^T \geq 0$. In fact (9) is sometimes called the right polar decomposition. The left polar decomposition is $M = U\Sigma V^T = U\Sigma U^T\ UV^T$. Do not confuse the polar decomposition with the so called QR factorisation. $M = QR$ where $Q \in O(n)$ and R upper triangular, which arises e.g. in the Gram-Schmidt orthogonalisation procedure.