DNA Modelling Course
Exercise Session 2
Summer 2006 Part 2

Introduction

In the exercise session 1 we considered various properties of the Link integral

$$Lk(C_1,C_2) = \frac{1}{4\pi}\int_{C_1}\int_{C_2}\frac{(y(\sigma)-x(s))\cdot(y'(\sigma)\times x'(s))}{|y(\sigma)-x(s)|^3}d\sigma\, ds$$

where C_1 (or $y(\sigma)$) and C_2 (or $x(s)$) are two non-intersecting curves.
The Writhe integral of a single curve C_1 is defined to be the double integral

$$Wr(C_{1,}) = \frac{1}{4\pi}\int_{C_1}\int_{C_1}\frac{(x(\sigma)-x(s))\cdot(x'(\sigma)\times x'(s))}{|x(\sigma)-x(s)|^3}d\sigma\, ds.$$

We denote the integrand of the Writhe integral by I_{Wr}:

$$I_{Wr}(\sigma,s) = \frac{(x(\sigma)-x(s))\cdot(x'(\sigma)\times x'(s))}{|x(\sigma)-x(s)|^3}.$$

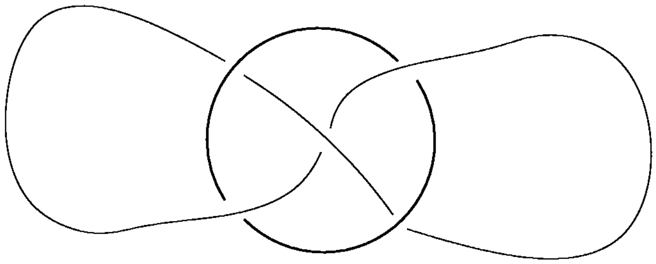
In Problem 1 you have to compute the Whitehead link, and in Problem 2 we consider some properties of the Writhe integral. We discussed the importance and applications of this integral in the classes. Notice that both the denominator and numerator of the Writhe integral vanish along the diagonal $s=\sigma$, so some analysis is necessary to show that the integrand is integrable (see Problem 2a.)). This difficulty does not arise for the Link double integral because $y(\sigma)\neq x(s)$ for all σ, s by the hypothesis that $C_1 \cap C_2 = \emptyset$.

Problem 1: The linking number of the Whitehead link.

Calculate the link of the Whitehead link shown in the figure below. Use the method of counting signed crossings.

Problem 2: Properties and Computations of Writhe Integral

a.) Show that the Writhe integral is even about $s=\sigma$, that is $I_{Wr}(\sigma,s) = I_{Wr}(s,\sigma)$ for all $\sigma, s \in C_1$. (Easy, but important.)



b.) Show that $\lim_{|\sigma-s|\to 0} I_{Wr}(\sigma, s) = 0$.

Hint: First expand $x(\sigma)$ in a finite Taylor series

$$x(\sigma) = x(s) + (\sigma - s)x'(s) + \frac{1}{2}(\sigma - s)^2\hat{x}'' \tag{0.1}$$

$$x(\sigma) = x(s) + (\sigma - s)x'(s) + \frac{1}{2}(\sigma - s)^2 x''(s) + \frac{1}{6}(\sigma - s)^3\hat{x}''' \tag{0.2}$$

and

$$x'(\sigma) = x'(s) + (\sigma - s)x''(s) + \frac{1}{2}(\sigma - s)^2\bar{x}''' \tag{0.3}$$

where x' denotes the derivative of the vector function x with respect to s, and $\hat{x}''$, $\hat{x}'''$ and $\bar{x}'''$ denote the vectors of the finite truncation terms,

$$\hat{x}'' = \begin{pmatrix} x_1''(\xi_1) \\ x_2''(\xi_2) \\ x_3''(\xi_3) \end{pmatrix}, \qquad \hat{x}''' = \begin{pmatrix} x_1'''(\psi_1) \\ x_2'''(\psi_2) \\ x_3'''(\psi_3) \end{pmatrix}, \qquad \bar{x}''' = \begin{pmatrix} x_1'''(\chi_1) \\ x_2'''(\chi_2) \\ x_3'''(\chi_3) \end{pmatrix}$$

with $|\xi_i - s| \le |\sigma - s|$, $|\psi_i - s| \le |\sigma - s|$ and $|\chi_i - s| \le |\sigma - s|$ for all $i = 1, 2, 3$.

Calculate the numerator using (0.2) and (0.3). In simplifying the numerator there is substantial cancellation, to survive is a term $O(|\sigma - s|^4)$.

Calculate the denominator of $I_{Wr}(\sigma, s)$ using (0.1), you get

$$|\sigma - s|^3 \underbrace{|x'(s) + \frac{1}{2}(\sigma - s)x''(\xi)|^3}$$

The braced term in the denominator is positive (assuming s is a smooth parameterization so that x' is a non-zero vector, for example a unit vector if s is an arc-length parameterization so that the braced term is greater than 1/2 for $|\sigma - s|$ sufficiently small).

Then the denominator behaves as $|\sigma - s|^3$, while the numerator behaves as $|\sigma - s|^4$. Therefore $I_{Wr} \to 0$ as $\sigma \to s$.

c.) Show that the Writhe of a curve is invariant under

- translations $x \mapsto x + a,\ a \in \mathbb{R}^3$
- rotations $x \mapsto Rx,\ RR^t = I,\ |R| = 1$
- dilations $x \mapsto \lambda x,\ \lambda > 0$

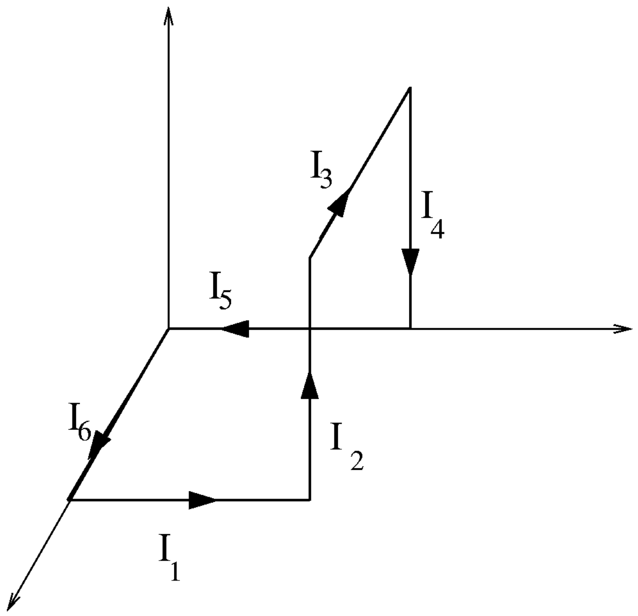
of the curve.

d.) Show that the Writhe of the reflection of a curve in a plane (i.e. the mirror image of the curve) is the negative of the Writhe of the curve.

$$\begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} \mapsto \begin{pmatrix} x_1 \\ x_2 \\ -x_3 \end{pmatrix}$$

e.) Show that the integrand of the Writhe vanishes pointwise for a planar (non-self-intersecting) curve, so that the Writhe of a planar curve is zero (which is necessary by Problem 2d.)).

f.) Just by looking at this figure, you can immediately tell the Writhe of this curve. (Keep in mind the previous exercises.)



g.) Calculate the Writhe of the curve consisting of four straight parts connecting the points $(1, 0, 0)$, $(0, 1, 0)$, $(0, 0, 0)$ and $(1, 0, 1)$, as in figure . (You may use Maple to evaluate integrals.)