DNA Modelling Course
Exercise Session 3
Summer 2006 Part 2

Problem 1: Writhe of a curve on a sphere.

Show that the Writhe of a non-self-intersecting closed curve lying on a sphere is zero. In particular Writhe vanishing is not a sufficient condition for a curve to be planar.

Hint: Use $Lk = Tw + Wr$ and explicitly construct a zero link, pointwise zero twist, $u_3 = 0$, closed framing. A direct proof seems hard, see Problem 3 for another approach.

Note: the Writhe integrand need *not* vanish pointwise as for a planar curve.

Problem 2: Inversion in a sphere.

We consider inversions in spheres for the proof of Problem 3. If you like, just use the results and switch directly to Problem 3.

Conformal geometry considers objects that are preserved by the Möbius group. The Möbius group is the set of finite compositions of inversions in a sphere or in a plane.
The inversion $I_\sigma : \mathbb{R}^3 \cup \{\infty\} \to \mathbb{R}^3 \cup \{\infty\}$ in a sphere σ of radius a and center $\boldsymbol{c}$ is given by

$$I_\sigma : \boldsymbol{x} \to \boldsymbol{c} + \frac{a^2}{|\boldsymbol{x}-\boldsymbol{c}|^2}(\boldsymbol{x}-\boldsymbol{c}), \quad I_\sigma(\boldsymbol{c}) = \infty, \quad I_\sigma(\infty) = \boldsymbol{c}.$$

(a) Check the relation $|\boldsymbol{x}-\boldsymbol{c}|\,|I_\sigma(\boldsymbol{x})-\boldsymbol{c}| = a^2$ for $\boldsymbol{x} \in \mathbb{R}^3 \backslash \{\boldsymbol{c}\}$, in particular this inversion leaves the sphere σ invariant.

(b) Show that it maps circles and spheres (including straight lines and planes) into circles and spheres.

(c) Inversions in spheres preserve angles between curves, that is the angle between the tangents of the curves.

Problem 3: Total Twist under inversions.

(See T. F. Banchoff and J. H. White, The behavior of the total twist and self-linking number of a closed space curve under inversions, Math. Scand. 36 (1975), 254-262.)

The Total Twist of a curve $\boldsymbol{x}$ with a frame $(\boldsymbol{d}_1, \boldsymbol{d}_2, \boldsymbol{d}_3)$ is given by

$$Tw(\boldsymbol{x}, \boldsymbol{d}_1) = \frac{1}{2\pi} \int_0^L u_3 \, ds.$$

Consider the change in Total Twist under inversions of the curve $\boldsymbol{x}$ in a sphere σ of radius a and center $\boldsymbol{c}$, i.e. under the mapping

$$I_\sigma : \boldsymbol{x} \to \boldsymbol{c} + \frac{a^2}{|\boldsymbol{x} - \boldsymbol{c}|^2}(\boldsymbol{x} - \boldsymbol{c}).$$

Let $\boldsymbol{d}$ be a unit vector field along $\boldsymbol{x}$. The oriented straight line generated by $\boldsymbol{d}$ is mapped by I_σ into an oriented circle. The tangent to this circle will be denoted by $\bar{\boldsymbol{d}}$. Then

$$\bar{\boldsymbol{d}} = \boldsymbol{d} - \frac{2\boldsymbol{d} \cdot (\boldsymbol{x} - \boldsymbol{c})}{(\boldsymbol{x} - \boldsymbol{c}) \cdot (\boldsymbol{x} - \boldsymbol{c})}(\boldsymbol{x} - \boldsymbol{c}).$$

Use this fact to prove $Tw(I_\sigma \boldsymbol{x}, \bar{\boldsymbol{d}}_1) = -Tw(\boldsymbol{x}, \boldsymbol{d}_1)$.

From the Calugareanu-Fuller-White Theorem we conclude that for a closed curve $Wr(I_\sigma \boldsymbol{x}) = -Wr(\boldsymbol{x})$ so that the Writhe of any curve on the sphere must vanish.

Problem 4: General form of the Darboux vector of an adapted framing of a given curve.

Given a smooth curve $\boldsymbol{r}(s)$ and a function $u_3(s)$, where s is the arclength parameter, show that

$$\boldsymbol{\xi}' = (u_3(s)\boldsymbol{r}' + \boldsymbol{r}' \times \boldsymbol{r}'') \times \boldsymbol{\xi} \tag{0.1}$$

with initial condition

$$\boldsymbol{\xi}(0) \cdot \boldsymbol{r}'(0) = 0, \quad |\boldsymbol{\xi}(0)|^2 = 1 \tag{0.2}$$

generates an orthonormal framing $(\boldsymbol{\xi}, (\boldsymbol{r}' \times \boldsymbol{\xi}), \boldsymbol{r}')$ of $\boldsymbol{r}(s)$.
Verify and calculate

1. That $|\boldsymbol{\xi}(0)|^2 = 1 \implies |\boldsymbol{\xi}(s)|^2 = 1 \quad \forall s$.
2. That $\boldsymbol{\xi}(0) \cdot \boldsymbol{r}'(0) = 0 \implies \boldsymbol{\xi}(s) \cdot \boldsymbol{r}'(s) = 0 \quad \forall s$.
3. Now, by picking an initial value of $\boldsymbol{\xi}(0)$ satisfying (0.2) we have an orthonormal frame $(\boldsymbol{\xi}, (\boldsymbol{r}' \times \boldsymbol{\xi}), \boldsymbol{r}')$ of $\boldsymbol{r}(s)$. What is the Darboux vector?
4. Note that $y = \boldsymbol{r}'$ and $z = \boldsymbol{r}' \times \boldsymbol{\xi}$ must be two other solutions of (0.1). Check!
5. If $\boldsymbol{r}'' \neq 0$, what is the Darboux vector in terms of the Serret-Frenet frame?

The facts (a) and (b) say that $|\boldsymbol{\xi}|^2$ and $\boldsymbol{\xi} \cdot \boldsymbol{r}'$ are integrals of the system (0.1).

Problem 5: Self-link of a curve on a sphere.

Let $\boldsymbol{r}(s)$ be a closed curve without inflection points on a sphere. Let s be the arclength parameter. The Self-link is the Link using the Serret-Frenet frame, i.e. $SLk(\boldsymbol{r}) = Lk(\boldsymbol{r}, \boldsymbol{r} + \varepsilon \boldsymbol{n})$ for ε small ($\boldsymbol{n}$ is the principal normal).

1. Show that the Self-Link of $\boldsymbol{r}(s)$ is zero.

2. Let the curve $\boldsymbol{r}(s)$ be non self-intersecting. Show that the Twist $Tw(\boldsymbol{r})$ is zero.