DNA Modeling Course
Exercise Session 4
Summer 2006 Part 2

Problem 1: Fenchel transforms

Let $\boldsymbol{x}, \boldsymbol{y} \in \mathbb{R}^n$ and consider a smooth function $\psi(\boldsymbol{x}) \in \mathbb{R}$ with Fenchel transform $\psi^*(\boldsymbol{y}) \in \mathbb{R}$ defined as

$$\psi^*(\boldsymbol{y}) = \max_{\boldsymbol{x} \in R^n} \left[\boldsymbol{y} \cdot \boldsymbol{x} - \psi(\boldsymbol{x})\right]. \tag{1.1}$$

(a) Show that the Fenchel transform of $\psi(\boldsymbol{x}) = \frac{1}{2}\boldsymbol{x} \cdot \boldsymbol{A}\boldsymbol{x} + \xi \cdot \boldsymbol{x}$ is

$$\psi^*(\boldsymbol{y}) = \frac{1}{2}(\boldsymbol{y} - \xi) \cdot \boldsymbol{A}^{-1}(\boldsymbol{y} - \xi),$$

where ξ is a vector and $\boldsymbol{A}$ is an $n \times n$ symmetric, positive definite, constant matrix. What happens when $\boldsymbol{A}$ is symmetric, $\det(A) \neq 0$, but $\boldsymbol{A} < 0$?

(b) Given any $\boldsymbol{a} \in \mathbb{R}^n$ consider the function $\bar{\psi}(\boldsymbol{x}) = \psi(\boldsymbol{x} + \boldsymbol{a})$. Show that $\bar{\psi}^*(\boldsymbol{y}) = \psi^*(\boldsymbol{y}) - \boldsymbol{y} \cdot \boldsymbol{a}$.

(c) Given any $\boldsymbol{M} \in \mathbb{R}^{n \times n}$ with $\det[\boldsymbol{M}] \neq 0$ consider the function $\bar{\psi}(\boldsymbol{x}) = \psi(\boldsymbol{M}\boldsymbol{x})$. Show that $\bar{\psi}^*(\boldsymbol{y}) = \psi^*(\boldsymbol{M}^{-T}\boldsymbol{y})$.

(d) Given any $\boldsymbol{a} \in \mathbb{R}^n$ and $\boldsymbol{M} \in \mathbb{R}^{n \times n}$ with $\boldsymbol{M}^{-1} = \boldsymbol{M}^T$ (so $\boldsymbol{M}$ is orthogonal) consider the function $\bar{\psi}(\boldsymbol{x}) = \psi(\boldsymbol{M}\boldsymbol{x} + \boldsymbol{a})$. Express the Fenchel transform of $\bar{\psi}$ in terms of the transform of ψ.

Problem 2: Hamiltonian form

Compute the Lagrangian and Hamiltonian forms of Euler-Lagrangwe equations and natural BC for the following functionals

(a)

$$\int_0^1 K_2(s)(\phi' - \hat{u}_2)^2/2 \,+\, \lambda \cos\phi \, ds,$$

(b)

$$\int_0^1 W(v_1 - \hat{v}_1(s), v_3 - \hat{v}_3(s), \phi'(s) - \hat{u}_2(s), s) \, ds \,+\, \lambda z(1),$$

where the functional in **(a)** corresponds to the energy of an inextensible and unshearable rod, whereas the functional in **(b)** corresponds to the energy of planar deformation of shearable and extensible rod with

$$\begin{aligned}
\boldsymbol{d}_1 &= (\cos\phi, 0, -\sin\phi)^T \\
\boldsymbol{d}_3 &= (\sin\phi, 0, \cos\phi)^T \\
\boldsymbol{r}' &= (\boldsymbol{x}', 0, \boldsymbol{z}') \\
\boldsymbol{v}_1 &= \boldsymbol{r}' \cdot \boldsymbol{d}_1 \\
\boldsymbol{v}_3 &= \boldsymbol{r}' \cdot \boldsymbol{d}_3 \\
\boldsymbol{u}_2 &= \phi'
\end{aligned}$$