DNA Modelling Course
Exercise Session 4
Summer 2006 Part 2

SOLUTIONS

Problem 1: Fenchel transforms

(a)

$$\psi^*(\boldsymbol{y}) = \max_{\boldsymbol{x}}(\boldsymbol{y}\cdot\boldsymbol{x} - \frac{1}{2}\boldsymbol{x}\cdot\boldsymbol{A}\boldsymbol{x} + \xi\cdot\boldsymbol{x}).$$

The maximum occurs when $\boldsymbol{y} = \boldsymbol{A}\boldsymbol{x} + \xi$ or $\boldsymbol{x} = \boldsymbol{A}^{-1}(\boldsymbol{y} - \xi)$. Therefore

$$\begin{aligned} psi^*(\boldsymbol{y}) &= \boldsymbol{y}\cdot\boldsymbol{A}^{-1}(\boldsymbol{y}-\xi) - \frac{1}{2}\boldsymbol{A}^{-1}(\boldsymbol{y}-\xi)\cdot\boldsymbol{A}\boldsymbol{A}^{-1}(\boldsymbol{y}-\xi) \\ &= \boldsymbol{y}\cdot\boldsymbol{A}^{-1}(\boldsymbol{y}-\xi) - \frac{1}{2}\boldsymbol{A}^{-1}(\boldsymbol{y}-\xi)\cdot(\boldsymbol{y}-\xi) \\ &= \boldsymbol{y}\cdot\boldsymbol{A}^{-1}(\boldsymbol{y}-\xi) - \frac{1}{2}(\boldsymbol{y}-\xi)\cdot\boldsymbol{A}^{-1}(\boldsymbol{y}-\xi) \\ &= \frac{1}{2}(\boldsymbol{y}-\xi)\cdot\boldsymbol{A}^{-1}(\boldsymbol{y}-\xi) \end{aligned}$$

(b)

$$\begin{aligned} \bar{\psi}^*(\boldsymbol{y}) &= \max_{\boldsymbol{x}}(\boldsymbol{y}\cdot\boldsymbol{x} - \psi(\boldsymbol{x}-\boldsymbol{a})) \\ &= \max_{\bar{\boldsymbol{x}}}(\boldsymbol{y}\cdot(\bar{\boldsymbol{x}}+\boldsymbol{a}) - \psi(\bar{\boldsymbol{x}})) \\ &= \max_{\bar{\boldsymbol{x}}}(\boldsymbol{y}\cdot\bar{\boldsymbol{x}} - \psi(\bar{\boldsymbol{x}})) + \boldsymbol{y}\cdot\boldsymbol{a} \\ &= \psi^*(\boldsymbol{y}) + \boldsymbol{y}\cdot\boldsymbol{a}. \end{aligned}$$

(c)

$$\begin{aligned} \bar{\psi}^*(\boldsymbol{y}) &= \max_{\boldsymbol{x}}(\boldsymbol{y}\cdot\boldsymbol{x} - \bar{\psi}(\boldsymbol{x})) \\ &= \max_{\boldsymbol{x}}(\boldsymbol{y}\cdot\boldsymbol{x} - \psi(\boldsymbol{M}\boldsymbol{x})) \\ &= \max_{\boldsymbol{z}}(\boldsymbol{y}\cdot\boldsymbol{M}^{-1}\boldsymbol{z} - \psi(\boldsymbol{z})) \\ &= \max_{\boldsymbol{z}}(\boldsymbol{M}^{-T}\boldsymbol{y}\cdot\boldsymbol{z} - \psi(\boldsymbol{z})) \\ &= \psi^*(\boldsymbol{M}^{-T}\boldsymbol{y}). \end{aligned}$$

(d)

$$\begin{aligned}
\bar{\psi}^*(\boldsymbol{y}) &= \max_{\boldsymbol{x}}(\boldsymbol{y}\cdot\boldsymbol{x} - \bar{\psi}(\boldsymbol{x})) \\
&= \max_{\boldsymbol{x}}(\boldsymbol{y}\cdot\boldsymbol{x} - \psi(\boldsymbol{M}\boldsymbol{x}+\boldsymbol{a})) \\
&= \max_{\boldsymbol{z}}(\boldsymbol{y}\cdot\boldsymbol{M}^{-1}(\boldsymbol{z}-\boldsymbol{a}) - \psi(\boldsymbol{z})) \\
&= \max_{\boldsymbol{z}}(\boldsymbol{M}\boldsymbol{y}\cdot(\boldsymbol{z}-\boldsymbol{a}) - \psi(\boldsymbol{z})) \\
&= \max_{\boldsymbol{z}}(\boldsymbol{M}\boldsymbol{y}\cdot\boldsymbol{z} - \psi(\boldsymbol{z})) \; - \; \boldsymbol{M}\boldsymbol{y}\cdot\boldsymbol{a} \\
&= \psi^*(\boldsymbol{M}\boldsymbol{y}) \; - \; \boldsymbol{M}\boldsymbol{y}\cdot\boldsymbol{a}.
\end{aligned}$$

Problem 2: Hamiltonian form

Refer the following pages of Chapter 4 of the course notes:
page 50 (example 1)
page 56 (example 4)
page 58 (example 1')
page 61 (example 4').