DNA Modelling Course
Exercise Session 3
Summer 2006 Part 1

Balance Laws

$$\begin{aligned} \boldsymbol{n}_s + \boldsymbol{f} &= \boldsymbol{0} \\ \boldsymbol{m}_s + \boldsymbol{r}_s \times \boldsymbol{n} + \boldsymbol{\tau} &= \boldsymbol{0} \end{aligned}$$

$\boldsymbol{f}$ and $\boldsymbol{\tau}$ are the distributed external force and moment loadings on the rod for s in the open interval $(0, L)$. Note that $\boldsymbol{n}(0), \boldsymbol{n}(L), \boldsymbol{m}(0), \boldsymbol{m}(L)$ are given by boundary conditions, or "end loadings".

1 Configurations and Equilibria of an Extensible Shearable Rod

Let $\{\boldsymbol{e}_1, \boldsymbol{e}_2, \boldsymbol{e}_3\}$ be a fixed basis for $\mathbb{R}^3$ and consider a rod modeled on the interval $[0, L]$ with configuration

$$\left.\begin{aligned} \boldsymbol{r}(s) &= (1+\epsilon)(\cos s\, \boldsymbol{e}_1 + \sin s\, \boldsymbol{e}_2) \\ \boldsymbol{d}_1(s) &= -\cos s\, \boldsymbol{e}_1 - \sin s\, \boldsymbol{e}_2 \\ \boldsymbol{d}_2(s) &= \boldsymbol{e}_3 \\ \boldsymbol{d}_3(s) &= -\sin s\, \boldsymbol{e}_1 + \cos s\, \boldsymbol{e}_2. \end{aligned}\right\} \tag{1.1}$$

for $|\epsilon|$ small.

Kinematics

1. Assume that $s \in [0, L]$ with $L = 2\pi$.

(a) Sketch the rod and calculate the components of $\boldsymbol{v}$ and $\boldsymbol{u}$ in the variable frame $\{\boldsymbol{d}_i\}$.

(b) Compute the unit quaternion describing the frame.

Balance Laws

2. Assume now that $s \in [0, L]$ with $L = \pi$. The rod has a straight natural configuration given by

$$\begin{cases} \hat{\boldsymbol{r}}(s) = (\frac{\pi}{2} - s)\boldsymbol{e}_1 \\ \hat{\boldsymbol{d}}_1(s) = -\boldsymbol{e}_2 \\ \hat{\boldsymbol{d}}_2(s) = \boldsymbol{e}_3 \\ \hat{\boldsymbol{d}}_3(s) = -\boldsymbol{e}_1 \end{cases} \tag{1.2}$$

(a) What are $\boldsymbol{n}$ and $\boldsymbol{m}$ for the linear constitutive elastic law

$$\left.\begin{array}{rcl} \mathsf{n}_i &=& \mathsf{G}_{ij}[\mathsf{v}_j - \hat{\mathsf{v}}_j] \\ \mathsf{m}_i &=& \mathsf{K}_{ij}[\mathsf{u}_j - \hat{\mathsf{u}}_j] \end{array}\right\} \tag{1.3}$$

where

$$\mathsf{G} = \begin{pmatrix} \mathsf{G}_1 & 0 & 0 \\ 0 & \mathsf{G}_2 & 0 \\ 0 & 0 & \mathsf{G}_3 \end{pmatrix} \quad \text{and} \quad \mathsf{K} = \begin{pmatrix} \mathsf{K}_1 & 0 & 0 \\ 0 & \mathsf{K}_2 & 0 \\ 0 & 0 & \mathsf{K}_3 \end{pmatrix}$$

are constant ?

(b) Assuming $\boldsymbol{\tau} \equiv \mathbf{0}$, for what distributed load $\boldsymbol{f}$, and for what end loads $\boldsymbol{g}$ (force) and $\boldsymbol{h}$ (moment) at $s = 0$ and π is the configuration $\{\boldsymbol{r}, \boldsymbol{d}_i\}$ an equilibrium configuration?

2 Helical curves

Given the parametrized helix

$$(x, y, z) = (R\cos t, R\sin t, Pt) \tag{2.1}$$

where R and P are two real numbers, compute

(a) the arclength parametrization;

(b) the Frenet frame, curvature and torsion;

(c) the Darboux vector.

(d) Compute the quaternion parametrization $q(t)$ of the Frenet frame [N B T] for $t = \pi$. Why is the analogous expression complicated when $t \neq \pi$?

3 Transformation of Cross Products

(a) Show that for any vectors $\boldsymbol{a}, \boldsymbol{b} \in \mathbb{R}^3$, and for any non-singular matrix $\boldsymbol{A} \in \mathbb{R}^{3\times 3}$,

$$\boldsymbol{A}\boldsymbol{a} \times \boldsymbol{A}\boldsymbol{b} = \det(\boldsymbol{A})\boldsymbol{A}^{-T}(\boldsymbol{a} \times \boldsymbol{b}), \qquad \boldsymbol{A}^{-T} := (\boldsymbol{A}^{-1})^T = (\boldsymbol{A}^T)^{-1}.$$

(b) From the previous result, show that for any proper orthogonal matrix $\boldsymbol{Q} \in \mathrm{SO}(3)$

$$\boldsymbol{Q}\boldsymbol{a} \times \boldsymbol{Q}\boldsymbol{b} = \boldsymbol{Q}(\boldsymbol{a} \times \boldsymbol{b})$$

Hint: use the fact that if $\boldsymbol{x}$, $\boldsymbol{y}$ and $\boldsymbol{z}$ are vectors in $\mathbb{R}^3$, then the triple product $\boldsymbol{x} \cdot (\boldsymbol{y} \times \boldsymbol{z})$ equals the determinant $\det([\boldsymbol{x}, \boldsymbol{y}, \boldsymbol{z}])$.