DNA Modelling Course
Exercise Session 4
Summer 2006 Part 1

In the first two exercises we contrast an *untwisted* circular rod configuration with a uniformly *twisted* straight configuration. This should clarify the distinction between twisted and untwisted configurations.

> Note: A configuration $\{\boldsymbol{r}, \boldsymbol{d}_i\}$ of an inextensible/unshearable rod with $r' = \boldsymbol{d}_3$ is called *untwisted* if $\boldsymbol{u} \cdot \boldsymbol{d}_3 \equiv 0$, and called *twisted* otherwise. If $\boldsymbol{u} \cdot \boldsymbol{d}_3 \equiv c \neq 0$ the configuration is called *uniformly twisted*.

1 Equilibria of an Inextensible and Unshearable Rod without Twist

Consider an inextensible and unshearable rod modeled on the interval $[0, L]$ with a straight reference configuration defined by

$$\hat{\boldsymbol{r}}(s) = s\boldsymbol{e}_3 \quad \text{and} \quad \hat{\boldsymbol{d}}_i(s) = \boldsymbol{e}_i.$$

Assume the rod obeys a linear elastic material law with a constant, diagonal stiffness matrix K, and assume no external distributed loads.

(a) Show that an untwisted, circular configuration with $\boldsymbol{d}_2(s) = \boldsymbol{e}_2$ is an equilibrium configuration of the rod. What end loads $\boldsymbol{g}$ and $\boldsymbol{h}$ can produce this equilibrium?

(b) Show that an untwisted, circular configuration with $\boldsymbol{d}_1(s) = \boldsymbol{e}_1$ is an equilibrium configuration of the rod. What end loads $\boldsymbol{g}$ and $\boldsymbol{h}$ can produce this equilibrium? Is this case identical to the previous one?

(c) Are there other untwisted, circular equilibrium configurations when $\mathsf{K}_1 \neq \mathsf{K}_2$? What about when $\mathsf{K}_1 = \mathsf{K}_2$?

2 Equilibria of an Inextensible and Unshearable Rod with Twist

In the same conditions described by the previous exercise,

(a) Show that a straight, uniformly twisted configuration defined by

$$\left.\begin{aligned} \boldsymbol{r}(s) &= s\boldsymbol{e}_3 \\ \boldsymbol{d}_1(s) &= \cos(2\pi\vartheta s/L)\boldsymbol{e}_1 + \sin(2\pi\vartheta s/L)\boldsymbol{e}_2 \\ \boldsymbol{d}_2(s) &= -\sin(2\pi\vartheta s/L)\boldsymbol{e}_1 + \cos(2\pi\vartheta s/L)\boldsymbol{e}_2 \\ \boldsymbol{d}_3(s) &= \boldsymbol{e}_3, \end{aligned}\right\} \tag{2.1}$$

where $\vartheta \in I\!R$ is a constant that specifies the twist rate, is an equilibrium configuration of the rod. What end loads $\boldsymbol{g}$ and $\boldsymbol{h}$ can produce this equilibrium?

(b) Show that a uniformly twisted circle is an equilibrium if, and only if, $\mathsf{K}_1 = \mathsf{K}_2$.

3 Computation of Unit Quaternion

Compute the unit quaternion describing the frame $\{\boldsymbol{d}_i(s)\}_i$, where for every $s \in [0, 2\pi]$

$$\boldsymbol{d}_1(s) = \begin{pmatrix} \cos(s) \\ \sin(s) \\ 0 \end{pmatrix}, \qquad \boldsymbol{d}_2(s) = \begin{pmatrix} -\sin(s) \\ \cos(s) \\ 0 \end{pmatrix}, \qquad \boldsymbol{d}_3(s) = \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix}.$$