DNA Modelling Course
Exercise Session 5
Summer 2006 Part 1

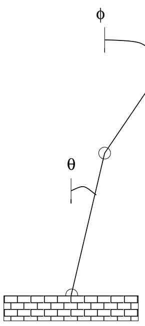
The purpose of this session is to generalize what was seen in class for a single rigid bar to a system of two rigid bars.

1 Equilibrium configurations for a system of two rigid bars

Consider a system of two rigid bars as shown in the figure below. A configuration of the system is described by two angle variables $(\theta, \phi) \in \mathbb{R}^2$, and the system has an energy function $E : \mathbb{R}^2 \times \mathbb{R} \to \mathbb{R}$ of the form

$$E(\theta, \phi; \lambda) = \frac{1}{2}\theta^2 + \frac{1}{2}(\theta - \phi)^2 + \lambda(\cos\theta + \cos\phi) \tag{1.1}$$

where $\lambda > 0$ is a parameter representing an external downward force.



Equilibrium configurations for this system correspond to critical points of E.

(a) The equilibrium equations are obtained by setting the gradient of $E(\theta, \phi; \lambda)$ to zero for fixed λ. Write the equilibrium equations and verify that $(\theta_0, \phi_0) = (0, 0)$ is an equilibrium configuration for any $\lambda > 0$.

(b) Along the equilibrium configuration $(0, 0; \lambda)$, apply the implicit function theorem to verify that for each $\lambda \neq \lambda_0^i$ where $\lambda_0^1 = (3 - \sqrt{5})/2$ and $\lambda_0^2 = (3 + \sqrt{5})/2$, there exists a unique solution branch in a neighborhood of the trivial solution. Conversely, at the bifurcations points λ_0^i we expect to find new, non-trivial solutions close to the trivial one.

(c) At the fixed values $\lambda = \lambda_0^i = (3 \mp \sqrt{5})/2$ determine the null space of Jacobian matrix used in the previuos step.

2 Stability

For a function in one dimension $\boldsymbol{E} : \mathbb{R} \to \mathbb{R}$ critical points are solutions of $\boldsymbol{E}'(x) = 0$. Then one can further determine the type of the critical point (maximum, minimum or saddle) by finding the sign of the second derivative $\boldsymbol{E}''(x)$ at each critical point. The analogous result for a function $\boldsymbol{E} : \mathbb{R}^n \to \mathbb{R}$ of n variables is to determine the signs of the eigenvalues of the Hessian matrix.

In the case of $n = 2$, if the critical point is a local minima, i.e. both eigenvalues are positive, then the equilibrium is said to be *stable*. On the other hand, if the critical point is a saddle or a maxima, i.e. one or more eigenvalues are negative, the equilibrium is said to be *unstable*. Finally, if the critical point is degenerate (i.e. the Hessian is singular, i.e. there is a zero eigenvalue), then the equilibrium is also called degenerate (it might or might not be a stable).

(a) Compute the Hessian hess$[E] : \mathbb{R}^2 \to \mathbb{R}^{2\times 2}$ defined by

$$\text{hess}[E](\theta, \phi) = \begin{pmatrix} \frac{\partial^2 E}{\partial \theta^2} & \frac{\partial^2 E}{\partial \phi \partial \theta} \\ \frac{\partial^2 E}{\partial \theta \partial \phi} & \frac{\partial^2 E}{\partial \phi^2} \end{pmatrix} \tag{2.1}$$

and determine the stability of the equilibrium $(\theta, \phi, \lambda) = (0, 0, \lambda)$ as a function of $\lambda > 0$. (For this part you need only the *signs* of the eigenvalues, which for a real 2×2 matrix can be determined from its trace and determinant.) Note that the Hessian is a function of the solution at which is is evaluated. Study how the stability (i.e. signs of the eigenvalues) changes along the trivial solution $(0, 0, \lambda)$ for varying λ. Is there a relation between bifurcation and change in stability? Is there a relation between points where the stability changes and degenerate points where the implicit function theorem cannot be applied?