DNA Modelling Course
Exercise Session 6
Summer 2006 Part 1

1 Finding shape of bifurcation branch

Consider Exercise 1 from the last session (Session 5). The bifurcation points $\lambda_0^{(i)}$ $(i = 1, 2)$ are where new, non-trivial equilibrium solutions can appear. To study how these new solutions behave in a neighbourhood of the trivial solution, we expand them in terms of a small parameter ε. More precisely, we consider perturbation expansions about each bifurcation point $\lambda_0^{(i)}$, namely

$$\begin{aligned}
\lambda_\varepsilon &= \lambda_0^{(i)} + \varepsilon\lambda_1^{(i)} + \varepsilon^2\lambda_2^{(i)} + \cdots \\
\theta_\varepsilon &= \theta_0 + \varepsilon\theta_1^{(i)} + \varepsilon^2\theta_2^{(i)} + \varepsilon^3\theta_3^{(i)} + \cdots \\
\phi_\varepsilon &= \phi_0 + \varepsilon\phi_1^{(i)} + \varepsilon^2\phi_2^{(i)} + \varepsilon^3\phi_3^{(i)} + \cdots .
\end{aligned} \tag{1.1}$$

We then substitute these expansions into the equilibrium equation

$$\boldsymbol{F}(\theta, \phi, \lambda) = \mathbf{0} \tag{1.2}$$

where $\boldsymbol{F} : \mathbb{R}^2 \times \mathbb{R}_+ \to \mathbb{R}^2$ is defined as

$$\boldsymbol{F}(\theta, \phi, \lambda) = \begin{cases} F_1(\theta, \phi, \lambda) &= \frac{\partial E}{\partial \theta}, \\ F_2(\theta, \phi, \lambda) &= \frac{\partial E}{\partial \phi}. \end{cases}$$

and then expand in powers of ε, demanding that each (i.e. at each order) coefficient vanish. For each i, this leads to the family of equations

$$\boldsymbol{A}_j \boldsymbol{v}_{j+1} = \boldsymbol{b}_j, \qquad j = 0, 1, 2, \ldots \tag{1.3}$$

where $\boldsymbol{v}_k = (\theta_k, \phi_k)$. Here $\boldsymbol{A}_n$ and $\boldsymbol{b}_n$ may depend upon θ_m, ϕ_m and λ_m for $m \leq n$.

(a) For each bifurcation point, determine θ_1, ϕ_1, λ_1 and λ_2 by $\boldsymbol{F}_1 = 0$, $\boldsymbol{F}_2 = 0$ and $\boldsymbol{F}_3 = 0$, where

$$\boldsymbol{F}_i = \frac{1}{i!} \frac{d^i}{d\varepsilon^i} \boldsymbol{F}(\theta_\varepsilon, \phi_\varepsilon, \lambda_\varepsilon)\Big|_{\varepsilon=0} \tag{1.4}$$

(Here you must use the fact that $\boldsymbol{A}\boldsymbol{v} = \boldsymbol{b}$ is solvable if and only if $\boldsymbol{b}$ is orthogonal to the null space of $\boldsymbol{A}^T$. Also, note that the solution for the perturbation variables (θ_1, ϕ_1) is unique up to a multiplicative constant, which you may set to unity. Why?)

(b) Equilibrium configurations (θ, ϕ) under a given load $\lambda > 0$ define a set in (θ, ϕ, λ)-space. What does the projection of this set look like in the (θ, λ)-plane? (Make a sketch, by hand or with the aid of a computer.)

2 Stability

Consider Exercise 2 of previous Session 5.

(a) How does the stability of the equilibrium configurations vary along the bifurcation branch sketched in part (1b) of this exercises Session? (From Problem 2 of last exercises Session you know the stability along the trivial solution. Thus, the real question here is to investigate stability along the new solutions, which you know in a neighborhood of the bifurcation points. To obtain stability information for the new solutions, just substitute their perturbation expansion in the Hessian and determine the signs of the eigenvalues for small ε.) Is there a relation between the shape of the bifurcation branch and its stability?