DNA Modelling Course
Exercise Session 7
Summer 2006 Part 1

Introduction

Consider an inextensible, unshearable rod of unit length in the plane defined by the orthonormal basis $(\boldsymbol{e}_1, \boldsymbol{e}_3)$. Let the curve $[r_1(s), r_3(s)]$ represent the centerline as a function of arclength $s \in [0,1]$. Let $\phi(s)$ be the clockwise angle between $\boldsymbol{e}_3$ and the tangent $[r_1'(s), r_3'(s)]$ to the rod. Let $[N_1(s), N_3(s)]$ and $m_2(s)$ be the force and the amplitude of the moment in the rod.

As shown in class the equations for the equilibrium of the rod are

$$N_1' = 0, \tag{0.1}$$
$$N_3' = 0, \tag{0.2}$$
$$\phi' = \frac{m_2}{K_2} + \hat{u}_2, \tag{0.3}$$
$$m_2' = -N_1 \cos\phi + N_3 \sin\phi, \tag{0.4}$$
$$r_1' = \sin\phi, \tag{0.5}$$
$$r_3' = \cos\phi, \tag{0.6}$$

where we assume that the unstressed strain $\hat{u}_2$ is constant along the rod and the bending stiffness K_2 is a linear function of arclength that depends on the parameter κ:

$$K_2(s) = 0.5 + \kappa(0.5 - s). \tag{0.7}$$

We will constrain the parameter κ in the interval $(-1, 1)$ so that $K_2(s) > 0$, for $s \in [0,1]$, and the right-hand-side of equation (0.3) is always defined.

We consider the strut boundary conditions

$$N_1(1) = \nu, \tag{0.8}$$
$$N_3(1) = -\lambda, \tag{0.9}$$
$$\phi(0) = 0, \tag{0.10}$$
$$m_2(1) = 0, \tag{0.11}$$
$$r_1(0) = 0, \tag{0.12}$$
$$r_3(0) = 0. \tag{0.13}$$

When the parameters have the values $\lambda = 0$, $\nu = 0$, $\hat{u}_2 = 0$ and $\kappa = 0$, the boundary value problem (0.1)-(0.13) has the trivial solution

$$N_1(s) = 0, \quad (0.14)$$
$$N_3(s) = 0, \quad (0.15)$$
$$\phi(s) = 0, \quad (0.16)$$
$$m_2(s) = 0, \quad (0.17)$$
$$r_1(s) = 0, \quad (0.18)$$
$$r_3(s) = s. \quad (0.19)$$

Starting from this trivial solution, the software AUTO can increment the value of any one of the parameters and numerically compute the corresponding solutions of the boundary value problem. VBM is a graphical interface to AUTO.

Practice

(a) To get some experience with VBM/AUTO, follow the steps described in the *VBM Visualization Tutorial* and the *VBM Computation Tutorial* [1].

(b) The equations used in *VBM Computation Tutorial* correspond to the boundary value problem (0.1)-(0.13). Use VBM/AUTO to do continuation in the parameter λ and construct the $0 \leq \lambda \leq 5$ portion of the bifurcation diagram in the $[\lambda, r_1(1)]$ plane for $\nu = 0$, $\hat{u}_2 = 0$ and $\kappa = 0$. Use a *data probe* of the type *numerical data at one point* to check the values of the parameters.

(c) Starting from a point in the bifurcation diagram of part (b), do continuation in the parameter $\hat{u}_2$, and then, starting from the point where $\hat{u}_2 = 0.4$ (to find this point, use a *data probe* of the type *numerical data at one point*), use continuation in the parameter λ to construct the $0 \leq \lambda \leq 5$ portion of the bifurcation diagram in the $[\lambda, r_1(1)]$ plane for $\nu = 0$, $\hat{u}_2 = 0.4$ and $\kappa = 0$. In order to get all the branches of the $\hat{u}_2 = 0.4$ bifurcation diagram, you have to execute this procedure at least twice starting from different points (preferably at both ends of the computed λ-section, i.e. on different sides of the pitchfork bifurcation point) in the $\hat{u}_2 = 0$ bifurcation diagram.
(By the way, if you want to compute in the minus direction, choose the -1 direction in the panel with the start parameters. In the next window the number of steps must be negative!)

Note that the bifurcation diagrams obtained in parts (b) and (c) are similar to the bifurcation diagrams of the 1 degree-of-freedom strut presented in class.

[1] `http://lcvmwww.epfl.ch/VBM/index.html, then select Documentation/Tutorial`

Exercise

(a) Verify that the trivial straight solution (0.14)-(0.19) satisfies the boundary value problem (0.1)-(0.13) for $\lambda \in \mathbb{R}$, $\kappa \in (-1, 1)$, $\nu = 0$ and $\hat{u}_2 = 0$.

(b) Use VBM/AUTO to compute the first and second bifurcation points in the $[\lambda, r_1(1)]$ plane for $\nu = 0$, $\hat{u}_2 = 0$ and $\kappa = -0.5$. What are the values of λ at these bifurcation points ? Same question for $\kappa = 0$ and for $\kappa = 0.5$.

(c) The value of λ at the first bifurcation point is called *buckling load* and corresponds to the lowest vertical force at which buckling of the rod begins. How does the buckling load vary as κ goes from -0.5 to 0 to 0.5 ? How can you explain this physically ?

(d) Use the General Auto Solution Tool (under 'Add Data Probe') to draw and compare the shapes of the buckled rod corresponding to $\kappa = -0.5$, $\kappa = 0$ and $\kappa = 0.5$ for $\lambda = 1.7$, $\nu = 0$ and $\hat{u}_2 = 0$ (on the first non-trivial branch).

(e) Compute the $[\lambda, r_1(1)]$ bifurcation diagram for $\nu \neq 0$, $\hat{u}_2 = 0$ and $\kappa = 0$. Do the same for $\nu = 0$ and $\hat{u}_2 \neq 0$. Notice that the symmetry breaking from the bifurcation diagram with $\hat{u}_2 = 0$ and $\nu = 0$ can happen in either direction depending on the sign of the perturbation in ν or $\hat{u}_2$. For fixed $\hat{u}_2 \neq 0$ vary ν such that the symmetry breaking switches from "up" to "down". Do you get a symmetric pitchfork for certain non-zero values of ν and $\hat{u}_2$ or do you get something else? (Hint: notice that the problem with $\nu = \hat{u}_2 = 0$ has the discrete symmetry that if $\phi(s)$ is a solution then $-\phi(s)$ is also a solution, but the preturbed problem does not have this symmetry).
You may find it easier to restart with a 'empty' bifurcation diagram. Maybe it is clearer if you don't go too far in ν direction, i.e. if you don't take too large value of ν.

How to start VBM?

Preparation for VBM

1. Create a working directory with your name, e.g. by

```
mkdir VBM_dir
cd VBM_dir
```

2. Type

```
source ~jwalter/vbm_env_copy.csh
```

This script sets some environment variables and copies some data necessary for the exercices in your current directory.

3. You can start the demo by

```
VBM.py -f demo1
```

Note : Don't forget to press ENTER or DRAW anytime you change a value. VBM ignores it otherwise!

Note 2 : Rules on the IMA computer usage can be found at http://ima.epfl.ch/it/regles.html