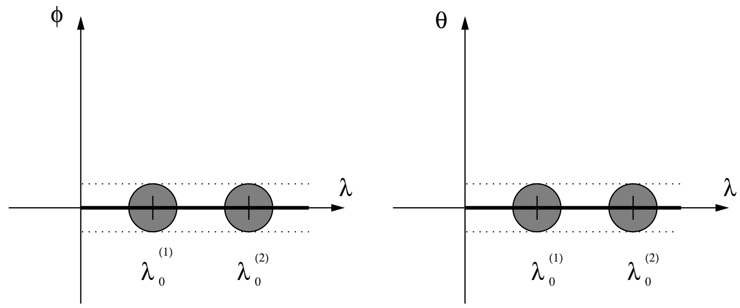
DNA Modelling Course
Exercise Session 6
Summer 2006 Part 1

SOLUTIONS

1 Finding shape of bifurcation branch

From Exercise 1 of last Session 5, we expect to find new, non-trivial solutions close to the trivial one $(\theta_0, \phi_0) = (0, 0)$ when $\lambda = \lambda_0^{(i)}$ $(i = 1, 2)$ as shown in the following figure. To study how these new solutions behave in a neighborhood of



the trivial solution, we expand them in terms of a small parameter ε. That is, for each bifurcation point $\lambda_0^{(i)}$, we consider perturbation expansions of the form

$$\left.\begin{aligned} \lambda_\varepsilon &= \lambda_0^{(i)} + \varepsilon\lambda_1^{(i)} + \varepsilon^2\lambda_2^{(i)} + \cdots \\ \theta_\varepsilon &= \theta_0 + \varepsilon\theta_1^{(i)} + \varepsilon^2\theta_2^{(i)} + \varepsilon^3\theta_3^{(i)} + \cdots \\ \phi_\varepsilon &= \phi_0 + \varepsilon\phi_1^{(i)} + \varepsilon^2\phi_2^{(i)} + \varepsilon^3\phi_3^{(i)} + \cdots . \end{aligned}\right\} \tag{1.1}$$

We then substitute these expansions into the equilibrium equation

$$\boldsymbol{F}(\theta, \phi, \lambda) = \boldsymbol{0}, \tag{1.2}$$

expand in powers of ε, and demand that each coefficient vanish. For each i, this leads to the family of equations

$$\boldsymbol{A}_j \boldsymbol{v}_{j+1} = \boldsymbol{b}_j, \qquad j = 0, 1, 2, \ldots \tag{1.3}$$

where $\boldsymbol{v}_k = (\theta_k, \phi_k)$. Here $\boldsymbol{A}_n$ and $\boldsymbol{b}_n$ may depend upon θ_m, ϕ_m and λ_m for $m \leq n$. For the current problem we have

$$\boldsymbol{A}_j = \begin{pmatrix} 2 - \lambda_0 & -1 \\ -1 & 1 - \lambda_0 \end{pmatrix}, \qquad j = 0, 1, 2, \ldots \tag{1.4}$$

and

$$\boldsymbol{b}_0 = \left\{ \begin{array}{c} 0 \\ 0 \end{array} \right\}, \quad \boldsymbol{b}_1 = \lambda_1 \left\{ \begin{array}{c} \theta_1 \\ \phi_1 \end{array} \right\}, \quad \boldsymbol{b}_2 = \lambda_2 \left\{ \begin{array}{c} \theta_1 \\ \phi_1 \end{array} \right\} + \lambda_1 \left\{ \begin{array}{c} \theta_2 \\ \phi_2 \end{array} \right\} - \frac{\lambda_0}{6} \left\{ \begin{array}{c} \theta_1^3 \\ \phi_1^3 \end{array} \right\}, \cdots \tag{1.5}$$

(a) For each $i = 1, 2$, the perturbation variables $\boldsymbol{v}_1 = (\theta_1, \phi_1)$ satisfy the equation

$$\boldsymbol{A}_0 \boldsymbol{v}_1 = \boldsymbol{0} \tag{1.6}$$

so $\boldsymbol{v}_1 \in \mathcal{N}[\boldsymbol{A}_0]$, that is

$$\left\{ \begin{array}{c} \theta_1 \\ \phi_1 \end{array} \right\} = a \left\{ \begin{array}{c} 1 \\ 2 - \lambda_0 \end{array} \right\} \tag{1.7}$$

for some constant $a \neq 0$. (Without loss of generality we may take $a = 1$ or a so that $\theta_1^2 + \phi_1^2 = 1$, that is, we may absorb the arbitrary constant into the parameter ε.) The next equation in the family is

$$\boldsymbol{A}_1 \boldsymbol{v}_2 = \boldsymbol{b}_1 \quad \text{where} \quad \boldsymbol{A}_1 = \boldsymbol{A}_0, \quad \boldsymbol{b}_1 = \lambda_1 \boldsymbol{v}_1. \tag{1.8}$$

This equation is solvable for $\boldsymbol{v}_2$ if and only if $\boldsymbol{b}_1$ is orthogonal to $\mathcal{N}[\boldsymbol{A}_0^T] = \mathcal{N}[\boldsymbol{A}_0]$. Since

$$\boldsymbol{b}_1 = \lambda_1 \boldsymbol{v}_1 \quad \text{and} \quad \mathcal{N}[\boldsymbol{A}_0] = \text{span}\{\boldsymbol{v}_1\},$$

we must have $\lambda_1 = 0$. This implies $\boldsymbol{v}_2 \in \mathcal{N}[\boldsymbol{A}_0]$, but we do not need to determine $\boldsymbol{v}_2$. Knowing $\lambda_1 = 0$, the next equation in the family is

$$\boldsymbol{A}_2 \boldsymbol{v}_3 = \boldsymbol{b}_2 \quad \text{where} \quad \boldsymbol{A}_2 = \boldsymbol{A}_0, \quad \boldsymbol{b}_2 = \lambda_2 \left\{ \begin{array}{c} \theta_1 \\ \phi_1 \end{array} \right\} - \frac{\lambda_0}{6} \left\{ \begin{array}{c} \theta_1^3 \\ \phi_1^3 \end{array} \right\}. \tag{1.9}$$

This equation is solvable for $\boldsymbol{v}_3$ if and only if $\boldsymbol{b}_2$ is orthogonal to $\mathcal{N}[\boldsymbol{A}_0^T] = \mathcal{N}[\boldsymbol{A}_0]$. Since

$$\mathcal{N}[\boldsymbol{A}_0] = \text{span}\{(1, 2 - \lambda_0)\},$$

we must have

$$\boldsymbol{b}_2 = c \left\{ \begin{array}{c} 2 - \lambda_0 \\ -1 \end{array} \right\} \tag{1.10}$$

that is,

$$\lambda_2 \left\{ \begin{array}{c} \theta_1 \\ \phi_1 \end{array} \right\} - \frac{\lambda_0}{6} \left\{ \begin{array}{c} \theta_1^3 \\ \phi_1^3 \end{array} \right\} = c \left\{ \begin{array}{c} 2 - \lambda_0 \\ -1 \end{array} \right\} \tag{1.11}$$

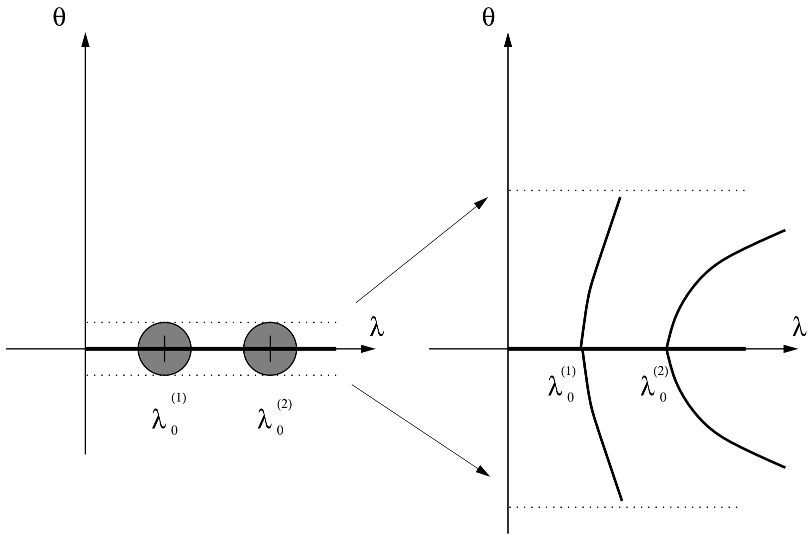
for some constant c. Equation (1.11) is actually a linear system in the unknowns λ_2 and c. Solving it we find $\lambda_2 = 1/(5 + \sqrt{5})$ for $i = 1$ and $\lambda_2 = 1/(5 - \sqrt{5})$ for $i = 2$. Thus, for the bifurcation point $\lambda_0^{(1)} = (3 - \sqrt{5})/2$ we have

$$\left. \begin{array}{rcl} \lambda_\varepsilon & = & \frac{3-\sqrt{5}}{2} + 0 + \varepsilon^2 \left[\frac{1}{5+\sqrt{5}} \right] + \cdots \\ \theta_\varepsilon & = & 0 + \varepsilon + \cdots \\ \phi_\varepsilon & = & 0 + \varepsilon \left[2 - \frac{3-\sqrt{5}}{2} \right] + \cdots, \end{array} \right\} \tag{1.12}$$

and for the bifurcation point $\lambda_0^{(2)} = (3+\sqrt{5})/2$ we have

$$\left.\begin{array}{rcl} \lambda_\varepsilon &=& \frac{3+\sqrt{5}}{2} + 0 + \varepsilon^2\left[\frac{1}{5-\sqrt{5}}\right] + \cdots \\ \theta_\varepsilon &=& 0 + \varepsilon + \cdots \\ \phi_\varepsilon &=& 0 + \varepsilon\left[2 - \frac{3+\sqrt{5}}{2}\right] + \cdots . \end{array}\right\} \tag{1.13}$$

(b) Using the result from part (a) we can get a picture of how equilibria look in the (θ, λ)-plane. By considering the perturbation expansions for small $|\varepsilon|$ we obtain the following figure. In particular, as $\lambda_1^{(i)} = 0$, $\lambda_2^{(i)} > 0$, both branches are supercritical, i.e. branches are symmetric and lie to the right of the bifurcation point.



2 Stability

(a) To determine the stability of the new solutions found in the previous exercise, just insert the perturbation expansions into hess$[E]$ and proceed as in the previous exercise. For each of the nonzero solution branches ($i = 1, 2$) we have

$$\begin{aligned} &\text{hess}[E](\theta_\varepsilon, \phi_\varepsilon, \lambda_\varepsilon) \\ &= \begin{pmatrix} 2-(\lambda_0+\varepsilon^2\lambda_2)(1-\varepsilon^2/2) & -1 \\ -1 & 1-(\lambda_0+\varepsilon^2\lambda_2)(1-\varepsilon^2(2-\lambda_0)^2/2) \end{pmatrix} \\ &\qquad + \mathcal{O}(\varepsilon^3). \end{aligned} \tag{2.1}$$

where

$$\lambda_0^{(1)} = (3-\sqrt{5})/2, \quad \lambda_2^{(1)} = 1/(5+\sqrt{5}),$$
$$\lambda_0^{(2)} = (3+\sqrt{5})/2, \quad \lambda_2^{(2)} = 1/(5-\sqrt{5}).$$

Note that it is the same number λ_2 which implies that the branch is a) supercritical (symmetric and lie to the right of the bifurcation point) and b) stable.

For small ε, the number of negative eigenvalues of hess$[E]$ is shown in the following figure.