DNA Modelling Course
Exercise Session 8
Summer 2006 Part 1

SOLUTIONS

Problem 1

(a) The equations are

$$\left\{\begin{array}{c} N_1' \\ N_3' \\ m_2' \\ \phi' \\ r_1' \\ r_3' \end{array}\right\} = \left\{\begin{array}{c} 0 \\ 0 \\ N_3\sin(\phi) - N_1\cos(\phi) \\ \frac{m_2}{K} \\ \sin(\phi) \\ \cos(\phi) \end{array}\right\}. \tag{1.1}$$

and clearly

$$u_0 = \left\{\begin{array}{c} N_{1,0} \\ N_{3,0} \\ m_{2,0} \\ \phi_0 \\ r_{1,0} \\ r_{3,0} \end{array}\right\} = \left\{\begin{array}{c} 0 \\ -\lambda \\ 0 \\ 0 \\ 0 \\ s \end{array}\right\}. \tag{1.2}$$

is a trivial solution for each $\lambda > 0$.

(b) Here we essentially search for other equilibria in a neighborhood of the trivial one u_0 with $\phi_0 \equiv 0$ (see (1.2)). To do this, we suppose there is a family of equilibria $u_\varepsilon = u_0 + \varepsilon\, u_1 + \varepsilon^2\, u_2 + \cdots$, parametrized by $|\varepsilon| \ll 1$, of the form

$$u_\varepsilon = \left\{\begin{array}{c} N_{1,\varepsilon} \\ N_{3,\varepsilon} \\ m_{2,\varepsilon} \\ \phi_\varepsilon \\ r_{1,\varepsilon} \\ r_{3,\varepsilon} \end{array}\right\} = \left\{\begin{array}{llll} N_{1,0} & +\,\varepsilon\, N_{1,1} & +\,\varepsilon^2\, N_{1,2} & +\cdots \\ N_{3,0} & +\,\varepsilon\, N_{3,1} & +\,\varepsilon^2\, N_{3,2} & +\cdots \\ m_{2,0} & +\,\varepsilon\, m_{2,1} & +\,\varepsilon^2\, m_{2,2} & +\cdots \\ \phi_0 & +\,\varepsilon\, \phi_1 & +\,\varepsilon^2\, \phi_2 & +\cdots \\ r_{1,0} & +\,\varepsilon\, r_{1,1} & +\,\varepsilon^2\, r_{1,2} & +\cdots \\ r_{3,0} & +\,\varepsilon\, r_{3,1} & +\,\varepsilon^2\, r_{3,2} & +\cdots \end{array}\right\}, \tag{1.3}$$

where $N_{1,0}$ is the first component of the trivial solution, $N_{1,1}$ is the linear term of the approximation, $N_{1,2}$ is the second order term of the approximation. For the other components respectively.

We insert u_ε into the equations (1.1) and gather terms of order ε. One way to do this is a Taylor expansion in ε. Another equal way is to regard (1.1) as a general system of the Form $\boldsymbol{F}[z(\varepsilon), \varepsilon] = 0$ ($\forall\, |\varepsilon| \ll 1$) and the linear terms are $\frac{d}{d\varepsilon}\Big|_{\varepsilon=0} \boldsymbol{F}[z(\varepsilon), \varepsilon] = \mathbf{0}$.

After much simplification (including use of boundary conditions) we get

$$\left\{ \begin{array}{c} K\phi_1'' + \lambda\phi_1 \\ \phi_1(0) \\ \phi_1(1) \end{array} \right\} = 0 \tag{1.4}$$

and we seek non-trivial solutions ϕ_1 to the linear equation (1.4). The most general solution of the differential equation in (1.4) is

$$\phi_1(s) = a\cos(\omega s) + b\sin(\omega s), \quad \omega = \sqrt{\frac{\lambda}{K}} > 0. \tag{1.5}$$

If ϕ_1 is to be non-trivial and satisfy the boundary conditions, then we must have $\omega = n\pi$ for $n = 1, 2, 3, \ldots$. The values of λ for which (1.4) admits non-trivial solutions are thus

$$\lambda = n^2\pi^2 K, \quad n = 1, 2, 3, \ldots \tag{1.6}$$

For these values of λ we expect new, non-trivial equilibrium solutions $u = (N_1, N_3, m_2, \phi, r_1, r_3)^t$ to bifurcate from the trivial solution u_0 (with $\phi_0 \equiv 0$).

For $\lambda = n^2\pi^2 K$ the solution is given by

$$u_\varepsilon = \left\{ \begin{array}{l} N_{1,\varepsilon} \\ N_{3,\varepsilon} \\ m_{2,\varepsilon} \\ \phi_\varepsilon \\ r_{1,\varepsilon} \\ r_{3,\varepsilon} \end{array} \right\} = \left\{ \begin{array}{l} 0 \\ -\lambda \\ \varepsilon\; n\pi K \cos(n\pi s) \\ \varepsilon\; \sin(n\pi s) \\ \frac{\varepsilon}{n\pi}(1 - \cos(n\pi s)) \\ s \end{array} \right\} + \; O(\varepsilon^2). \tag{1.7}$$

Hereby we used $b = 1$, all other cases may be absorbed into ε.

Problem 2

(a) As in Problem 1.

(b) Here we search for other equilibria in a neighborhood of the trivial one u_0 with $(\phi_0, N_{1,0}) \equiv (0, 0)$. As in Problem 1, we suppose there is a family of equilibria u_ε, parametrized by $|\varepsilon| \ll 1$, of the form (1.3).

Again we insert u_ε into the equations (1.1) and gather terms of order ε, but this time N_1 is unknown. After much simplification (including use of boundary conditions) we get therefore

$$\left\{\begin{array}{c} K\phi_1'' + \lambda\phi_1 + N_{1,1} \\ \int_0^1 \phi_1 \, ds \\ \phi_1(0) \\ \phi_1(1) \end{array}\right\} = \left\{\begin{array}{c} 0 \\ 0 \\ 0 \\ 0 \end{array}\right\}. \tag{2.1}$$

We seek non-trivial solutions $(\phi_1, N_{1,1})$ to (2.1). The most general solution of the differential equation in $(2.1)_1$ is

$$\phi_1(s) = a\cos(\omega s) + b\sin(\omega s) - \frac{N_{1,1}}{\lambda}, \quad \omega = \sqrt{\frac{\lambda}{K}} > 0. \tag{2.2}$$

Substituting (2.2) into the three equations $(2.1)_{2,3,4}$ leads to a homogeneous linear system of equations for a, b and $N_{1,1}$, namely

$$\begin{pmatrix} 1 & 0 & -1/\lambda \\ \cos\omega & \sin\omega & -1/\lambda \\ \sin\omega & 1-\cos\omega & -\omega/\lambda \end{pmatrix} \left\{\begin{array}{c} a \\ b \\ N_{1,1} \end{array}\right\} = \left\{\begin{array}{c} 0 \\ 0 \\ 0 \end{array}\right\}. \tag{2.3}$$

A non-trivial solution $(\phi_1, N_{1,1})$ to (2.1) can exist if and only if (2.3) admits non-trivial solutions. Thus we must have

$$\det\begin{pmatrix} 1 & 0 & -1/\lambda \\ \cos\omega & \sin\omega & -1/\lambda \\ \sin\omega & 1-\cos\omega & -\omega/\lambda \end{pmatrix} = 0 \qquad \Leftrightarrow \qquad \omega\sin\omega = 2[1-\cos\omega]. \tag{2.4}$$

For any $n = 1, 2, 3, \ldots$ note that $\omega = 2n\pi$ is a bifurcation point, and that there is another in each interval $2n\pi < \omega < (2n+1)\pi$. Given these bifurcation points, the constants a, b and $N_{1,1}$ can be determined from the null space of the matrix appearing in (2.3).

Note that a non-trivial solution $(\phi_1, N_{1,1})$ is defined only up to a multiplicative constant. W. l. o. g. we have $N_{1,1} = 0$ or $N_{1,1} = 1$, all other cases may be absorbed into ε. The solutions at $\lambda = 4n^2\pi^2 K$ correspond to the solutions with $N_{1,1} = 0$, the solutions at λ with $4n^2\pi^2 K < \lambda < (2n+1)^2\pi^2 K$ correspond to the solutions with $N_{1,1} = 1$.