5 Bifurcation Theory

In the modeling of physical systems one often considers a general equation of the form

$$\boldsymbol{F}(\boldsymbol{w}, \boldsymbol{\lambda}) = \boldsymbol{0} \tag{5.83}$$

where $\boldsymbol{\lambda} \in \boldsymbol{I}$ and $\boldsymbol{w} \in \boldsymbol{W}$. Here $\boldsymbol{I}$ is an interval in $\mathbb{R}$ and $\boldsymbol{W}$ is a subset of a given vector space. For example, when $\boldsymbol{W}$ is finite-dimensional, $\boldsymbol{F}(\boldsymbol{w}, \boldsymbol{\lambda})$ may be a set of algebraic equations, and when $\boldsymbol{W}$ is infinite-dimensional, $\boldsymbol{F}(\boldsymbol{w}, \boldsymbol{\lambda})$ may be a set of differential equations together with boundary conditions. A basic problem is to characterize the set of $\boldsymbol{w}$ satisfying (5.83) as the parameter $\boldsymbol{\lambda}$ is varied.

Example 5.1 *(Planar 1-DOF Strut) The equilibrium equations for a planar, one degree-of-freedom strut can be written in the form (5.83) with* $\boldsymbol{W} = \mathbb{R}$, *namely*

$$\boldsymbol{F}(\boldsymbol{\phi}, \boldsymbol{\lambda}) = \boldsymbol{\phi} - \boldsymbol{\beta} - \boldsymbol{\lambda} \sin \boldsymbol{\phi} = \boldsymbol{0}. \tag{5.84}$$

Here $\boldsymbol{\beta}$ *is a constant that defines the reference or unloaded configuration. For* $\boldsymbol{\beta} = \boldsymbol{0}$ *a portion of the solution set of (5.84) looks something like the figure on the left, and for* $\boldsymbol{\beta} > \boldsymbol{0}$ *a portion of the solution set looks something like the figure on the right.*

Remarks 5.1

1. Depending on the form of $\boldsymbol{F}(\boldsymbol{w}, \boldsymbol{\lambda})$, the solution set of (5.83) can be very complicated. We see that it can be *connected* in some cases, and it can be *disconnected* in others.

2. It is very rare that one can analytically determine solutions as in the above example. One must typically employ numerical procedures to solve (5.83). A common numerical procedure is *parameter continuation*.

By a (*linearized*) *bifurcation analysis* of (5.83) we mean an analysis in which one obtains *local* information on the solution set in a neighborhood of a known solution. For example, suppose we have a family or branch of solutions

$$\boldsymbol{F}(\boldsymbol{w_0}, \boldsymbol{\lambda}) = \boldsymbol{0}, \qquad \forall \boldsymbol{\lambda} \in \boldsymbol{I}, \tag{5.85}$$

for some $\boldsymbol{w_0} \in \boldsymbol{W}$. Then a bifurcation analysis would yield local information on the existence of other solutions in an arbitrarily small neighborhood of the branch $(\boldsymbol{w_0}, \boldsymbol{\lambda})$, $\boldsymbol{\lambda} \in \boldsymbol{I}$. For certain values of $\boldsymbol{\lambda}$ there could be no other solutions close to this branch, while for other values there could, as illustrated in Example 5.1 for the case $\boldsymbol{\beta} = \boldsymbol{0}$. Generally speaking, a (linearized) bifurcation analysis is based on an appropriate Implicit Function Theorem.

5.1 Basic Problem

Suppose we are given

1. Parameter-dependent equation. $\boldsymbol{F}(\boldsymbol{w}, \boldsymbol{\lambda}) = \boldsymbol{0}$, $\boldsymbol{w} \in \boldsymbol{W}$, $\boldsymbol{\lambda} \in \boldsymbol{I}$. Here $\boldsymbol{\lambda}$ is interpreted as the parameter.

2. Known solution branch. $(\boldsymbol{w_0}, \boldsymbol{\lambda})$, $\boldsymbol{\lambda} \in \boldsymbol{I}$.

To determine if there are other solutions in an arbitrarily small neighborhood of the known branch $(\boldsymbol{w_0}, \boldsymbol{\lambda})$, $\boldsymbol{\lambda} \in \boldsymbol{I}$, we

1. Expand $\boldsymbol{w}$ in neighborhood of $\boldsymbol{w_0}$

$$\boldsymbol{w_\varepsilon} = \boldsymbol{w_0} + \boldsymbol{\varepsilon w_1} + \boldsymbol{\varepsilon^2 w_2} + \cdots . \tag{5.86}$$

2. Search for non-trivial solutions $\boldsymbol{w_1}$ of the linearized equation

$$\left.\frac{\boldsymbol{d}}{\boldsymbol{d\varepsilon}}\right|_{\boldsymbol{\varepsilon=0}} \boldsymbol{F}(\boldsymbol{w_\varepsilon}, \boldsymbol{\lambda}) = \boldsymbol{A_0}(\boldsymbol{w_1}) = \boldsymbol{0}. \tag{5.87}$$

3. Identify bifurcation points $\boldsymbol{\lambda_0}$. By a *bifurcation point* we mean any $\boldsymbol{\lambda_0} \in \boldsymbol{I}$ for which (5.87) possesses a non-trivial solution.

At the bifurcation points $\boldsymbol{\lambda_0}$ we expect to find new solutions near the known branch $(\boldsymbol{w_0}, \boldsymbol{\lambda})$, $\boldsymbol{\lambda} \in \boldsymbol{I}$.

Remarks 5.2

1. A bifurcation point λ_0 is called *simple* if (5.87) possesses only one independent, non-trivial solution. Otherwise, λ_0 is called *non-simple*.

2. Roughly speaking, the number of independent, non-trivial solutions of (5.87) determines the dimension of the solution set in a neighborhood of (w_0, λ_0). For simple bifurcation points, we thus expect to find a new branch or curve of solutions.

3. In the neighborhood of a simple bifurcation pair (w_0, λ_0) we can develop an asymptotic expansion of the new solution branch. Parameterizing by $\varepsilon \in \mathbb{R}$, we expand the new branch $(w_\varepsilon, \lambda_\varepsilon)$ about (w_0, λ_0) as

$$\left.\begin{aligned} w_\varepsilon &= w_0 + \varepsilon w_1 + \varepsilon^2 w_2 + \cdots \\ \lambda_\varepsilon &= \lambda_0 + \varepsilon \lambda_1 + \varepsilon^2 \lambda_2 + \cdots . \end{aligned}\right\} \tag{5.88}$$

If $(w_\varepsilon, \lambda_\varepsilon)$ is to be a solution for all $|\varepsilon| \ll 1$, then

$$F(w_\varepsilon, \lambda_\varepsilon) = 0, \qquad \forall |\varepsilon| \ll 1, \tag{5.89}$$

which implies

$$\frac{d^n}{d\varepsilon^n}\bigg|_{\varepsilon=0} F(w_\varepsilon, \lambda_\varepsilon) = 0 \qquad n = 1, 2, 3, \ldots . \tag{5.90}$$

From (5.90) we get a sequence of linear equations for the perturbation variables w_n, λ_n $(n = 1, 2, 3, \ldots)$.

5.2 Planar Strut

5.2.1 Bifurcation Analysis

Consider a planar, inextensible, unshearable elastic rod described by an angle function $\phi(s) \in \mathbb{R}$, $s \in [0, 1]$, and subjected to a vertical force $\lambda > 0$. Assuming the rod is completely fixed at $s = 0$ (bottom), and that only the orientation is fixed at $s = 1$ (top), the equilibrium equations can be written as

$$\left.\begin{aligned} &K\phi'' + \lambda \sin\phi = 0, \quad 0 < s < 1 \\ &\phi(0) = 0 \\ &\phi(1) = 0 \end{aligned}\right\} \tag{5.91}$$

where $K > 0$ is a material constant.

Remarks 5.3

1. The equilibrium equations for the current system are just the first-order conditions for a standard Calculus of Variations problem for the energy functional

$$E(\phi) = \int_0^1 \mathcal{E}(s, \phi(s), \phi'(s))\, ds, \quad \mathcal{E}(\sigma, q, p) = \frac{1}{2}Kp^2 + \lambda \cos q, \tag{5.92}$$

subject to the imposed boundary conditions $\phi(0) = \phi(1) = 0$.

2. In the abstract notation of the previous section the equilibrium equations can be written as

$$F(\phi, \lambda) = \left\{ \begin{array}{c} K\phi'' + \lambda \sin \phi \\ \phi(0) \\ \phi(1) \end{array} \right\} = \left\{ \begin{array}{c} 0 \\ 0 \\ 0 \end{array} \right\}. \tag{5.93}$$

For the current system it is easy to see that (ϕ_0, λ), $\lambda > 0$, is a solution branch of (5.93) where $\phi_0(s) \equiv 0$. To perform a bifurcation analysis of this branch we linearize as in (5.87) to obtain the system

$$\frac{d}{d\varepsilon}\Big|_{\varepsilon=0} F(\phi_\varepsilon, \lambda) = \left\{ \begin{array}{c} K\phi_1'' + \lambda\phi_1 \\ \phi_1(0) \\ \phi_1(1) \end{array} \right\} = \left\{ \begin{array}{c} 0 \\ 0 \\ 0 \end{array} \right\}. \tag{5.94}$$

We next seek bifurcation points $\lambda > 0$ for which (5.94) possesses non-trivial solutions ϕ_1.

The most general solution of the differential equation in (5.94) is

$$\phi_1(s) = a \cos(\omega s) + b \sin(\omega s), \quad \omega = \sqrt{\frac{\lambda}{K}} > 0. \tag{5.95}$$

If ϕ_1 is to be non-trivial and satisfy the boundary conditions, then we must have $\omega = n\pi$ for $n = 1, 2, 3, \ldots$. The values of λ for which (5.94) admits non-trivial solutions are thus

$$\lambda = n^2\pi^2 K, \quad n = 1, 2, 3, \ldots\ , \tag{5.96}$$

and we note that these bifurcation points are simple. For these values of λ we expect new, non-trivial solution branches to bifurcate from the trivial branch (ϕ_0, λ), $\lambda > 0$.

5.2.2 Stability Analysis

The current problem has an underlying variational structure. That is, the equilibrium equations (5.91), or equivalently (5.93), can be identified as the first-order conditions for a standard Calculus of Variations problem. We can thus speak of the *stability* of an equilibrium ϕ as a function of the parameter $\lambda > 0$.

Recall that an equilibrium is said to be stable if it minimizes the energy functional $E(\phi)$ given in (5.92) over all admissible ϕ. For the given class of admissible functions, sufficient conditions for stability are

(i) the inequality $\mathcal{E}_{pp}(s, \phi, \phi') > 0$ holds along the equilibrium ϕ

(ii) the closed interval $[0, 1]$ contains no conjugate points.

If (i) holds, but there are conjugate points in the open interval $(0, 1)$, then the equilibrium is not a minimum and hence is unstable. Recall that conjugate points are defined using the Jacobi accessory equation together with boundary conditions that depend on the problem.

For the current problem, a conjugate point for an equilibrium ϕ is any $\tau \in (0, 1]$ for which the Jacobi equation (after simplification)

$$Qh - \frac{d}{ds}(Ph') = 0, \tag{5.97}$$

with boundary conditions

$$h(0) = 0, \qquad h(\tau) = 0 \tag{5.98}$$

possesses a non-trivial solution $h(s)$, $s \in [0, \tau]$, where

$$P = \frac{1}{2}\mathcal{E}_{pp}(s, \phi, \phi'), \qquad Q = \frac{1}{2}\left[\mathcal{E}_{qq}(s, \phi, \phi') - \frac{d}{ds}\mathcal{E}_{qp}(s, \phi, \phi')\right]. \tag{5.99}$$

For the trivial equilibria $\phi_0(s) \equiv 0$ an application of the conjugate point test leads to the conclusion that

$$\lambda < \pi^2 K \quad \Rightarrow \quad \text{stability}, \tag{5.100}$$

while

$$\lambda > \pi^2 K \quad \Rightarrow \quad \text{instability}. \tag{5.101}$$

The details of these calculations are left as an exercise.

Remark 5.4 For this and other variational problems note that the linearization (5.94) is connected to the second variation of the underlying functional, and is related to the Jacobi accessory equation and conjugate points. For example, note that a bifurcation point for the above problem gives rise to a conjugate point at $\tau = 1$.

5.3 3D Strut

Consider an inextensible and unshearable rod modeled on the interval $[0, 1]$ with a straight reference configuration defined by $\hat{r}(s) = s e_3$ and $\hat{d}_i(s) = e_i$, and subjected to a downward vertical force at one end with magnitude $\lambda > 0$.

Assuming the rod obeys a linear elastic material law with a constant, diagonal stiffness matrix $\mathsf{K} = \boldsymbol{diag}(\mathsf{K}_1, \mathsf{K}_2, \mathsf{K}_3)$, the equilibrium equations are

$$\left.\begin{array}{l} n' = 0 \\ m' + r' \times n = 0 \\ r' = d_3 \\ \\ m = \mathsf{K}_{ij} \mathsf{u}_j d_i \\ \mathsf{u}_i = \frac{1}{2} \varepsilon_{ijk} [d_j' \cdot d_k] \end{array}\right\}, \qquad \forall s \in (0, 1), \tag{5.102}$$

and the specific boundary conditions we want to consider are

$$\left.\begin{array}{lcll} r(0) & = & 0 & \\ d_i(0) & = & e_i & (i = 1, 2, 3) \\ r(1) \cdot e_i & = & 0 & (i = 1, 2) \\ d_i(1) & = & e_i & (i = 1, 2, 3) \\ n(1) \cdot e_3 & = & -\lambda & \end{array}\right\}. \tag{5.103}$$

For the current system it is easy to see that $(r_0, d_i^0, n_0, \lambda)$, $\lambda > 0$, is a solution branch of (5.102), (5.103) where

$$\left.\begin{array}{lcl} r_0(s) & = & s e_3 \\ d_i^0(s) & \equiv & e_i \\ n_0(s) & \equiv & -\lambda e_3 \end{array}\right\}, \qquad \forall \lambda > 0. \tag{5.104}$$

Our goal is to perform a bifurcation analysis in the two cases $\mathsf{K}_1 \neq \mathsf{K}_2$ and $\mathsf{K}_1 = \mathsf{K}_2$.

5.3.1 Linearization

To perform a bifurcation analysis of the trivial branch $(\boldsymbol{r}_0, \boldsymbol{d}_i^0, \boldsymbol{n}_0, \boldsymbol{\lambda})$, $\boldsymbol{\lambda} > \boldsymbol{0}$, we must linearize as in (5.87). To this end, we consider expansions of the form

$$\left.\begin{array}{rcl} \boldsymbol{r}_\varepsilon(s) & = & \boldsymbol{r}_0(s) + \varepsilon\boldsymbol{\eta}(s) + \cdots \\ \boldsymbol{d}_i^\varepsilon(s) & = & \boldsymbol{d}_i^0(s) + \varepsilon\boldsymbol{\xi}_i(s) + \cdots \\ \boldsymbol{n}_\varepsilon(s) & = & \boldsymbol{n}_0(s) + \varepsilon\boldsymbol{\gamma}(s) + \cdots \end{array}\right\}. \tag{5.105}$$

Note that since $\boldsymbol{d}_i^\varepsilon$ is a frame depending on $\varepsilon \in I\!\!R$, there exists a Darboux vector $\boldsymbol{\theta}^\varepsilon$ such that

$$\frac{d}{d\varepsilon}\boldsymbol{d}_i^\varepsilon = \boldsymbol{\theta}^\varepsilon \times \boldsymbol{d}_i^\varepsilon \tag{5.106}$$

and so we have

$$\boldsymbol{\xi}_i = \frac{d}{d\varepsilon}\bigg|_{\varepsilon=0} \boldsymbol{d}_i^\varepsilon = \boldsymbol{\theta} \times \boldsymbol{d}_i^0 \tag{5.107}$$

where we use the notation $\boldsymbol{\theta} = \boldsymbol{\theta}^0$. In what follows, we work with $\boldsymbol{\theta}$ rather than $\boldsymbol{\xi}_i$.

Remark 5.5 Note that an infinite number of terms are generally required in $(5.105)_2$ in order that the perturbed vectors $\boldsymbol{d}_i^\varepsilon$ indeed be an orthonormal frame. However, because only the first-order term enters the linearization, it is unnecessary to consider those of higher order.

Substituting (5.105) into (5.102), (5.103) we linearize to obtain the equations

$$\left.\begin{array}{l} \boldsymbol{\gamma}' = \boldsymbol{0} \\ (\mathsf{K}_{ij}\boldsymbol{\theta}_j'\boldsymbol{d}_i^0)' + \boldsymbol{\eta}' \times \boldsymbol{n}_0 + \boldsymbol{r}_0' \times \boldsymbol{\gamma} = \boldsymbol{0} \\ \boldsymbol{\eta}' = \boldsymbol{\theta} \times \boldsymbol{d}_3^0 \end{array}\right\}, \qquad \forall s \in (0,1), \tag{5.108}$$

where $\boldsymbol{\theta}_j = \boldsymbol{\theta} \cdot \boldsymbol{d}_j^0$, together with the boundary conditions

$$\left.\begin{array}{rcll} \boldsymbol{\eta}(0) & = & \boldsymbol{0} & \\ \boldsymbol{\theta}(0) & = & \boldsymbol{0} & \\ \boldsymbol{\eta}(1) \cdot \boldsymbol{e}_i & = & \boldsymbol{0} & (i = 1, 2) \\ \boldsymbol{\theta}(1) & = & \boldsymbol{0} & \\ \boldsymbol{\gamma}(1) \cdot \boldsymbol{e}_3 & = & \boldsymbol{0} & \end{array}\right\}. \tag{5.109}$$

In deriving the above equations one must use the fact that

$$\frac{d}{d\varepsilon}\bigg|_{\varepsilon=0} \boldsymbol{m}_\varepsilon = \mathsf{K}_{ij}\boldsymbol{\theta}_j'\boldsymbol{d}_i^0, \tag{5.110}$$

the proof of which is left as an exercise.

5.3.2 Simplifications

We next simplify the equations in (5.108), (5.109). To begin, we note from $(5.108)_1$ and $(5.109)_5$ that

$$\gamma(s) \equiv \alpha_1 e_1 + \alpha_2 e_2 \tag{5.111}$$

where α_1 and α_2 are constants. Next, we note that the variable η can be eliminated. Using $(5.108)_3$ and $(5.109)_1$ we have

$$\eta(s) = \int_0^s \theta(\tau) \times e_3 \, d\tau. \tag{5.112}$$

With the above equation we can convert the boundary conditions $(5.109)_3$ into conditions on θ. For example, $\eta(1) \cdot e_1 = 0$ implies

$$\int_0^1 \theta_2(s) \, ds = 0, \tag{5.113}$$

and so on.

The linearized equations (5.108) can be reduced to the form

$$\left. \begin{array}{l} \mathsf{K}_1\theta_1'' + \lambda\theta_1 - \alpha_2 = 0 \\ \mathsf{K}_2\theta_2'' + \lambda\theta_2 + \alpha_1 = 0 \\ \mathsf{K}_3\theta_3'' = 0 \end{array} \right\}, \qquad \forall s \in (0,1), \tag{5.114}$$

with side conditions

$$\left. \begin{array}{lcll} \theta_i(0) & = & 0 & (i = 1,2,3) \\ \theta_i(1) & = & 0 & (i = 1,2,3) \\ \int_0^1 \theta_1(s) \, ds & = & 0 & \\ \int_0^1 \theta_2(s) \, ds & = & 0 & \end{array} \right\}. \tag{5.115}$$

5.3.3 Bifurcation Points

We next seek values of $\lambda > 0$ for which the linearized system (5.114), (5.115) possesses non-trivial solutions $\theta_i(s)$, α_1 and α_2. Beginning with (5.114), note that the most general solution of these differential equations is

$$\left. \begin{array}{l} \theta_1(s) = a_1 \cos(\omega_1 s) + b_1 \sin(\omega_1 s) + \frac{\alpha_2}{\lambda} \\ \theta_2(s) = a_2 \cos(\omega_2 s) + b_2 \sin(\omega_2 s) - \frac{\alpha_1}{\lambda} \\ \theta_3(s) = a_3 s + b_3 \end{array} \right\} \tag{5.116}$$

where $\omega_i = \sqrt{\lambda/\mathsf{K}_i}$, and a_i, b_i are arbitrary constants.

The two conditions in (5.115) on $\boldsymbol{\theta_3}$ imply

$$\boldsymbol{a_3 = b_3 = 0} \qquad \Rightarrow \qquad \boldsymbol{\theta_3(s)} \text{ always trivial.} \tag{5.117}$$

The three conditions in (5.115) on $\boldsymbol{\theta_1}$ lead to a homogeneous linear system of equations for $\boldsymbol{a_1}$, $\boldsymbol{b_1}$ and $\boldsymbol{\alpha_2}$, namely

$$\begin{pmatrix} 1 & 0 & 1/\lambda \\ \cos\omega_1 & \sin\omega_1 & 1/\lambda \\ \sin\omega_1 & 1-\cos\omega_1 & \omega_1/\lambda \end{pmatrix} \begin{Bmatrix} a_1 \\ b_1 \\ \alpha_2 \end{Bmatrix} = \begin{Bmatrix} 0 \\ 0 \\ 0 \end{Bmatrix}. \tag{5.118}$$

Let $\boldsymbol{C_1}$ be the $\boldsymbol{3 \times 3}$ coefficient matrix in (5.118) and consider the set

$$\begin{aligned} \sigma_1 &= \{\lambda > 0 \mid \det C_1 = 0\} \\ &= \{\lambda > 0 \mid \omega_1 \sin\omega_1 = 2(1-\cos\omega_1)\} \\ &= \{\lambda > 0 \mid \lambda = 4n^2\pi^2\mathsf{K}_1,\ \lambda = \Lambda_1(n), \quad n = 1,2,3,\dots\}. \end{aligned} \tag{5.119}$$

Then there is a non-trivial solution $\boldsymbol{\theta_1}$ iff $\boldsymbol{\lambda \in \sigma_1}$.

Similarly, the three conditions in (5.115) on $\boldsymbol{\theta_2}$ lead to a homogeneous linear system of equations for $\boldsymbol{a_2}$, $\boldsymbol{b_2}$ and $\boldsymbol{\alpha_1}$, namely

$$\begin{pmatrix} 1 & 0 & -1/\lambda \\ \cos\omega_2 & \sin\omega_2 & -1/\lambda \\ \sin\omega_2 & 1-\cos\omega_2 & -\omega_2/\lambda \end{pmatrix} \begin{Bmatrix} a_2 \\ b_2 \\ \alpha_1 \end{Bmatrix} = \begin{Bmatrix} 0 \\ 0 \\ 0 \end{Bmatrix}. \tag{5.120}$$

Let $\boldsymbol{C_2}$ be the $\boldsymbol{3 \times 3}$ coefficient matrix in (5.120) and consider the set

$$\begin{aligned} \sigma_2 &= \{\lambda > 0 \mid \det C_2 = 0\} \\ &= \{\lambda > 0 \mid \omega_2 \sin\omega_2 = 2(1-\cos\omega_2)\} \\ &= \{\lambda > 0 \mid \lambda = 4n^2\pi^2\mathsf{K}_2,\ \lambda = \Lambda_2(n), \quad n = 1,2,3,\dots\}. \end{aligned} \tag{5.121}$$

Then there is a non-trivial solution $\boldsymbol{\theta_2}$ iff $\boldsymbol{\lambda \in \sigma_2}$.

The sets $\boldsymbol{\sigma_1}$ and $\boldsymbol{\sigma_2}$ thus characterize the bifurcation points for our problem. Whether or not these bifurcation points are simple depends on the values of K_1 and K_2.

5.3.4 Interpretations: $\mathsf{K}_1 \neq \mathsf{K}_2$

In the case $\mathsf{K}_1 \neq \mathsf{K}_2$ (and assuming that K_1 and K_2 are "non-resonant") we have $\boldsymbol{\sigma_1 \cap \sigma_2 = \emptyset}$, and so there are two distinct classes of simple bifurcation points. At a bifurcation point $\boldsymbol{\lambda \in \sigma_1}$

1. there is only one independent, non-trivial solution of the linearized equations, namely $\boldsymbol{\theta_1(s) \not\equiv 0}$, $\boldsymbol{\theta_2(s), \theta_3(s) \equiv 0}$

2. we expect one-dimensional branch of new solutions to bifurcate from the trivial branch

3. since $\boldsymbol{\theta_1(s) \not\equiv 0}$ and $\boldsymbol{\theta_2(s), \theta_3(s) \equiv 0}$, the new branch of solutions correspond to approximately planar configurations in the plane normal to $\boldsymbol{e_1}$.

Similarly, at a bifurcation point $\boldsymbol{\lambda} \in \boldsymbol{\sigma_2}$

1. there is only one independent, non-trivial solution of the linearized equations, namely $\boldsymbol{\theta_2(s) \not\equiv 0}$, $\boldsymbol{\theta_1(s), \theta_3(s) \equiv 0}$

2. we expect one-dimensional branch of new solutions to bifurcate from the trivial branch

3. since $\boldsymbol{\theta_2(s) \not\equiv 0}$ and $\boldsymbol{\theta_1(s), \theta_3(s) \equiv 0}$, the new branch of solutions correspond to approximately planar configurations in the plane normal to $\boldsymbol{e_2}$.

A bifurcation diagram for this problem would look something like the following. Note that the actual ordering of the bifurcation points depends on

the values of K_1 and K_2.

5.3.5 Interpretations: $\mathsf{K}_1 = \mathsf{K}_2$

In the case $\mathsf{K}_1 = \mathsf{K}_2$ there is only one class of bifurcation points characterized by the set $\boldsymbol{\sigma} = \boldsymbol{\sigma_1} = \boldsymbol{\sigma_2}$. At a bifurcation point $\boldsymbol{\lambda} \in \boldsymbol{\sigma}$

1. there are two independent, non-trivial solutions of the linearized equations: one with $\boldsymbol{\theta_1(s) \not\equiv 0}$, $\boldsymbol{\theta_2(s), \theta_3(s) \equiv 0}$, and the other with $\boldsymbol{\theta_2(s) \not\equiv 0}$, $\boldsymbol{\theta_1(s), \theta_3(s) \equiv 0}$

2. we expect a two-dimensional "surface" of new solutions to bifurcate from the trivial branch

3. each planar "slice" (identified by an angle $\boldsymbol{0 \leq \beta \leq 2\pi}$) is a branch of equilibria.

5.3.6 Symmetry-Breaking

The above analysis shows that bifurcation points along the trivial branch are not simple in the symmetric case $\mathsf{K}_1 = \mathsf{K}_2$. Here we show that, using known integrals of the equations (5.102), it is possible to single out a particular "slice" from the family of bifurcating equilibria and thus effectively reduce the problem to one with simple bifurcation points. While this can be done by just appending extra side conditions to the basic equations (5.102), (5.103), our interest here is to show that this can be done *without changing the number of equations*. The basic strategy is to modify the boundary conditions (5.103) to "break" the symmetry.

To begin, consider (5.102), (5.103) in the symmetric case $\mathsf{K}_1 = \mathsf{K}_2$. Let $\boldsymbol{\nu}$ be any unit vector in the $\{\boldsymbol{e}_1, \boldsymbol{e}_2\}$ plane and consider the *modified* boundary conditions

$$\left.\begin{array}{lcll} \boldsymbol{r}(\boldsymbol{0}) & = & \boldsymbol{0} & \\ (\boldsymbol{\nu}\cdot\boldsymbol{e}_1)[\boldsymbol{d}_2(\boldsymbol{0})\cdot\boldsymbol{\nu}] & = & (\boldsymbol{\nu}\cdot\boldsymbol{e}_2)[\boldsymbol{d}_1(\boldsymbol{0})\cdot\boldsymbol{\nu}] & \\ \boldsymbol{d}_3(\boldsymbol{0})\cdot\boldsymbol{\nu} & = & \boldsymbol{0} & \\ \boldsymbol{m}(\boldsymbol{0})\cdot\boldsymbol{\nu} & = & \boldsymbol{0} & \\ \boldsymbol{r}(\boldsymbol{1})\cdot\boldsymbol{e}_i & = & \boldsymbol{0} & (\boldsymbol{i}=\boldsymbol{1},\boldsymbol{2}) \\ \boldsymbol{d}_i(\boldsymbol{1}) & = & \boldsymbol{e}_i & (\boldsymbol{i}=\boldsymbol{1},\boldsymbol{2},\boldsymbol{3}) \\ \boldsymbol{n}(\boldsymbol{1})\cdot\boldsymbol{e}_3 & = & -\boldsymbol{\lambda} & \end{array}\right\}. \tag{5.122}$$

Claim 5.2 *The problem defined by (5.102), (5.122) has two classes of (isolated) solutions*

1. *those that satisfy the original problem (5.102), (5.103)*
2. *those that do not.*

Moreover, solutions of (5.102), (5.122) that satisfy (5.102), (5.103) and are planar lie in the $\{\boldsymbol{\nu}, \boldsymbol{e}_3\}$ plane (so $\boldsymbol{\nu}$ determines the "slice").

Proof. (Sketch) To begin, let $\{\boldsymbol{r}, \boldsymbol{d}_i, \boldsymbol{n}\}$ be a solution of (5.102), (5.122). Next, recall that the functions

$$\left.\begin{array}{lcl} \boldsymbol{N}(\boldsymbol{r},\boldsymbol{d}_i,\boldsymbol{n}) & = & \boldsymbol{n} \\ \boldsymbol{M}(\boldsymbol{r},\boldsymbol{d}_i,\boldsymbol{n}) & = & \boldsymbol{m}+\boldsymbol{r}\times\boldsymbol{n} \\ \boldsymbol{T}(\boldsymbol{r},\boldsymbol{d}_i,\boldsymbol{n}) & = & \boldsymbol{m}\cdot\boldsymbol{d}_3 \end{array}\right\} \tag{5.123}$$

are integrals for (5.102) in the symmetric case $\mathsf{K}_1 = \mathsf{K}_2$. Using these integrals together with $(5.122)_6$ we conclude that

$$\boldsymbol{m}(\mathbf{0}) \cdot \boldsymbol{d}_3(\mathbf{0}) = \boldsymbol{m}(\mathbf{0}) \cdot \boldsymbol{e}_3. \tag{5.124}$$

As long as $\boldsymbol{m}(\mathbf{0})$ is non-zero (as it is for non-trivial solutions), the above statements imply that $\boldsymbol{d}_3(\mathbf{0})$ and $\boldsymbol{e}_3$ lie in a cone with axis $\boldsymbol{m}(\mathbf{0})$.

In view of $(5.122)_3$, $(5.122)_4$ and the fact that $\boldsymbol{e}_3 \cdot \boldsymbol{\nu} = \mathbf{0}$, we also note that $\boldsymbol{d}_3(\mathbf{0})$, $\boldsymbol{e}_3$ and $\boldsymbol{m}(\mathbf{0})$ all lie in the plane normal to $\boldsymbol{\nu}$. This implies that either $\boldsymbol{d}_3(\mathbf{0}) = \boldsymbol{e}_3$, or $\boldsymbol{d}_3(\mathbf{0})$ is on the opposite side of the cone from $\boldsymbol{e}_3$. If $\boldsymbol{d}_3(\mathbf{0}) = \boldsymbol{e}_3$, then we deduce from $(5.122)_2$ that there are various isolated solutions for $\boldsymbol{d}_i(\mathbf{0})$ $(\boldsymbol{i} = \mathbf{1}, \mathbf{2})$, including the case $\boldsymbol{d}_i(\mathbf{0}) = \boldsymbol{e}_i$ $(\boldsymbol{i} = \mathbf{1}, \mathbf{2})$. Thus there are solutions $\{\boldsymbol{r}, \boldsymbol{d}_i, \boldsymbol{n}\}$ of (5.102), (5.122) that satisfy the original problem (5.102), (5.103). On the other hand, if $\boldsymbol{d}_3(\mathbf{0})$ is on the opposite side of the cone from $\boldsymbol{e}_3$, then clearly $\{\boldsymbol{r}, \boldsymbol{d}_i, \boldsymbol{n}\}$ does not satisfy the original problem. This is the basic idea, and the rest of the proof is left to the reader. ∎

To see that the modified problem (5.102), (5.122) has simple bifurcation points along the trivial branch, one would need to linearize this problem and repeat the analysis as before. This is a nice exercise.

Remark 5.6 Our motivation for symmetry-breaking is a computational one. For example, if one desires to numerically compute solutions of the above problem using a one-dimensional parameter continuation method, then simple bifurcation points are a must. Also, calculations are much simpler when working with a system possessing the same number of equations as unknowns. Moreover, while the symmetric case is "non-generic," it is sometimes advantageous to compute solutions of a "generic" problem beginning from the highly-connected solution set of an associated symmetric one.